# A bound for an integral

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May 6, 2012

## 1 Statement of the problem

Problem E2155 in the American Mathematical Monthly, December 1969, page 1142, poses the following problem which has been used to solve related problems. Suppose f(x) has a continuous $2n^{th}$ derivative on [a,b], that $|f^{2n}(x)| \leq M$ and that $f^{(r)}(a) = f^{(r)}(b) = 0$ for $r = 0, 1, 2 \dots n-1$. Show that $|\int_a^b f(x)dx| \leq \frac{(n!)^2 M}{(2n!)^2(2n+1)^2}(b-a)^{2n+1}$. Applications of this general solution to specific situations have figured in many problems in the American Mathematical Monthly over the years.

Hint: Let $g(x) = (x-a)^n(x-b)^n$ and consider $\int_a^b f(x)g^{(2n)}(x)dx$ and $\int_a^b g(x)f^{(2n)}(x)dx$

## 2 Solution

The solution given the journal was by Alberto Torchinsky of the University of Chicago and consists of 4 brief lines of proof without any intermediate steps ( see `http://www.jstor.org/stable/2317200` ). I fill in all the gaps in this article. Torchinsky takes f(x) as given in the assumptions and lets $g(x) = (x-a)^n(b-x)^n$. As you will see later I think this involves a typo given his statement of the solution but nothing turns on it. He claims that successive integration yields:

$$\int_a^b f(x)g^{(2n)}(x)dx = \int_a^b g(x)f^{(2n)}(x)dx = \int_a^b f(x)(x-a)^n(b-x)^n dx = \int_a^b (2n)!f(x)dx \tag{1}$$

He then states that:

$$|\int_a^b f(x)\,dx| \;=\; \frac{M}{(2n)!}(b-a)^{2n+1}\int_0^1 x^n(1-x)^n dx \;=\; \frac{M(b-a)^{2n+1}}{(2n)!}\frac{(n!)(n!)}{(2n+1)!} \qquad (2)$$

That's all there is to the solution. Now to fill in the gaps. Earlier I made the comment that there was a typo in the published solution. When you look at (1) it is clear that the solver is asserting that $g^{2n}(x) = (2n)!$ and this could not be the case if $g(x) = (x-a)^n(b-x)^n$. The reason is that $g(x) = (x-a)^n(b-x)^n = (-1)^n(x-a)^n(x-b)^n = (-1)^n[x^{2n} + \text{terms of lower power}]$. When this is differentiated 2n times you will get $(-1)^n(2n)!$ That this is so is verified later. However, nothing turns on this because we are taking absolute values so the factor of $(-1)^n$ is of no relevance. However in what follows I take $g(x) = (x-a)^n(x-b)^n$.

To establish the first equality in (1), namely, $\int_a^b f(x)g^{(2n)}(x)dx = \int_a^b g(x)f^{(2n)}(x)dx$ use integration by parts. When $n = 1$ the assertion is that $\int_a^b f(x)g^{(2)}(x)\,dx = \int_a^b g(x)f^{(2)}(x)\,dx$. Let $u = g(x)$ so that $du = g^{(1)}(x)\,dx$ and $v = f^{(1)}(x)$ with $dv = f^{(2)}(x)\,dx$. Then:

$$\int_a^b g(x)f^{(2)}(x)\,dx = \Big[g(x)f^{(1)}(x)\Big]_a^b - \int_a^b f^{(1)}(x)g^{(1)}(x)\,dx = -\int_a^b f^{(1)}(x)g^{(1)}(x)\,dx$$

noting that $g(a) = g(b) = 0$ (3)

Similarly, letting $u = f(x)$ so that $du = f^{(1)}(x)\,dx$ and $v = g^{(1)}(x)$ so that $dv = g^{(2)}(x)\,dx$ we have:

$$-\int_a^b f^{(1)}(x)g^{(1)}(x)\,dx = -\left\{\Big[f(x)g^{(1)}(x)\Big]_a^b - \int_a^b f(x)g^{(2)}(x)\,dx\right\} = \int_a^b f(x)g^{(2)}(x)\,dx$$

noting that $f(a) = f(b) = 0$ because one of the assumptions was that $f^{(0)}(a) = f^{(0)}(b) = 0$ (4)

Thus we have established that the assertion is true for $n = 1$. Now we suppose the assertion is true for any n and this is to the effect that:

$$\int_a^b f^{(2n)}(x)g(x)\,dx = \int_a^b g^{(2n)}(x)f(x)\,dx \text{ where g(x) has degree 2n} \tag{5}$$

Note that in the induction step, which involves $n+1$, $g(x) = (x-a)^{n+1}(x-b)^{n+1}$ has degree $2n+2$ but when it is differentiated twice it has degree $2n$.

So in $I = \int_a^b f^{(2n+2)}(x)g(x)\,dx$ let $u = f^{(2n+1)}(x)$ so that $du = f^{(2n+2)}(x)\,dx$ and $v = g(x)$ so that $dv = g^{(1)}(x)\,dx$. Hence:

$$I = \Big[g(x)f^{(2n+1)}(x)\Big]_a^b - \int_a^b f^{(2n+1)}(x)g^{(1)}(x)\,dx = -\int_a^b f^{(2n+1)}(x)g^{(1)}(x)\,dx$$
$$\text{noting that } g(a) = g(b) = 0 \tag{6}$$

Note here that we are implicitly assuming that $^{(2n+1)}(x)$ and $^{(2n+2)}(x)$ exist since the assumptions involve arbitrary n. Integrating by parts again we get:

$$I = -\left\{\Big[f^{(2n)}(x)g^{(1)}(x)\Big]_a^b - \int_a^b f^{(2n)}(x)g^{(2)}(x)\,dx\right\} = \int_a^b f^{(2n)}(x)g^{(2)}(x)\,dx$$
$$\text{since } g^{(1)}(a) = g^{(1)}(b) = 0 \text{ noting that } g(x) = (x-a)^{n+1}(x-b)^{n+1} \tag{7}$$

Now our inductive hypothesis ensures that $\int_a^b f^{(2n)}(x)g^{(2)}x)\,dx = \int_a^b f(x)g^{(2n+2)}(x)\,dx$ since $g^{(2)}(x)$ has degree $2n$. If $g(x) = x^{2n}+$ terms of lower powers then after 2n differentiations you simply get $g^{(2n)}(x) = (2n)!$ This can be verified inductively as follows.

Let $p_n(x) = \prod_{k=1}^{n}(x-a_k)$ so $p_n(x)$ has degree n with leading coefficient 1. Then $p_n^{(1)}(x) = 1!$ so the formula is true for $n = 1$. We assume as our induction step $p_n^n(x) = n!$. Then:

$$p_{n+1}(x) = (x - a_{n+1})p_n(x) \tag{8}$$

Now using Leibnitz's rule for differentiating products ( see Problem 17 in the Problem Set. Problem 23 could also be used) we have:

$$p_{n+1}^{(n+1)}(x) = \sum_{j=0}^{n+1} \binom{n+1}{j} \frac{d^{n+1-j}(x-a_{n+1})}{dx^{n+1-j}} \frac{d^j p_n(x)}{dx^j} = \binom{n+1}{n} \frac{d(x-a_{n+1})}{dx} \frac{d^n p_n(x)}{dx^n} \tag{9}$$

because all other terms vanish

$= (n+1)\, n!$ using the induction hypothesis
$= (n+1)!$

Hence the formula is true for $n+1$ and so for all n. Thus $g^{(2n)}(x) = (2n)!$

So finally, recalling (1), we have that $\int_a^b (2n)! f(x)\, dx = \int_a^b f^{(2n)}(x) g(x)\, dx$

ie

$$\int_a^b f(x)\, dx = \frac{1}{(2n)!} \int_a^b f^{(2n)}(x) g(x)\, dx \tag{10}$$

Taking absolute values:

$$\Big|\int_a^b f(x)\, dx\Big| \le \frac{1}{(2n)!} \int_a^b \Big|f^{(2n)}(x) g(x)\Big|\, dx \le \frac{M}{(2n)!} \int_a^b \Big|g(x)\Big|\, dx = \frac{M}{(2n)!} \int_a^b \Big|(x-a)^n (x-b)^n\Big|\, dx$$
$$= \frac{M}{(2n)!} \int_a^b (x-a)^n (b-x)^n\, dx \text{ noting that the integrand is non-negative on [a,b]} \tag{11}$$

Now we make the substitution $y = \frac{x-a}{b-a}$ in the last integral in (11) to get:

$$\frac{M}{(2n)!} \int_a^b (x-a)^n (b-x)^n\, dx = \frac{M(b-a)^{n+1}}{(2n)!} \int_0^1 y^n (b-a)^n (1-y)^n\, dx = \frac{M(b-a)^{2n+1}}{(2n)!} \int_0^1 y^n (1-y)^n\, dx$$
$$\text{noting that } b-x = (b-a)(1-y) \tag{12}$$

Now $\int_0^1 y^n (1-y)^n\, dx$ may be recognisable as a special case of the beta integral:

$$\int_0^1 t^p (1-t)^q\, dt = \frac{p!\, q!}{(p+q+1)!} \tag{13}$$

If (13) is not recognisable from probability/statistics courses you can use your "bare" hands to work out the final integral in (12) by integral reduction techniques. See the HSC Integral Reduction Formulas article here: `http://www.gotohaggstrom.com/HSC%20Integral%20Reduction%20Formulas.pdf`.

Thus following the inequalities through we find that:

$\left|\int_a^b f(x)\,dx\right| \leq \frac{M(b-a)^{2n+1}}{(2n)!}\,\frac{(n!)\,(n!)}{(2n+1)!}$ as claimed in the problem.

While there is nothing particularly deep in the components of this solution the insight was to define g(x) in the manner chosen. The details are fiddly and somewhat mehanical. A variation of that technique arises in various calculus of variations proofs as well. For instance in the proof of the necessary condition for an extremum there is the following lemma:

If g(x) is continuous in [a,b] and if $\int_a^b g(x)h(x)\,dx = 0$ for every function $h(x) \in \mathcal{C}[a,b]$ such that $h(a) = h(b) = 0$, then $g(x) = 0$ for all $x \in [a,b]$. In the proof of this lemma h(x) is defined as:
$h(x) = (x - x_1)(x - x_2)$for $x \in [x_1, x_2]$ and 0 otherwise. See I M Gelfand and S V Fomin "Calculus of Variations", Dover Publications, 1991, page 9.