Algebraic and geometric equivalence of the dot product

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September 9, 2016

The vector dot product of two vectors is usually defined in both geometric and algebraic terms. The geometric version which is favoured in physics contexts is as follows. For two vectors $\vec{a}$ and $\vec{b}$ the dot product is defined as :

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\,\cos\theta \tag{1}$$

where $|\vec{x}|$ is the magnitude of vector $\vec{x}$ ie $|\vec{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. The x_i are the components of $\vec{x}$ in the standard basis $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$. The angle between the two vectors is θ. Since $\cos\theta = \cos(-\theta)$ you can measure the angle from $\vec{a}$ to $\vec{b}$ or from $\vec{b}$ to $\vec{a}$.

The algebraic version of (1) is:

$$\vec{a} \cdot \vec{b} = \sum_{k=1}^{n} a_k b_k = a_1 b_1 + \cdots + a_n b_n \tag{2}$$

If $\vec{a}$ and $\vec{b}$ are represented by $\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}$ and $\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}$ then the direction cosines of the directions of $\vec{a}$ and $\vec{b}$ are respectively:

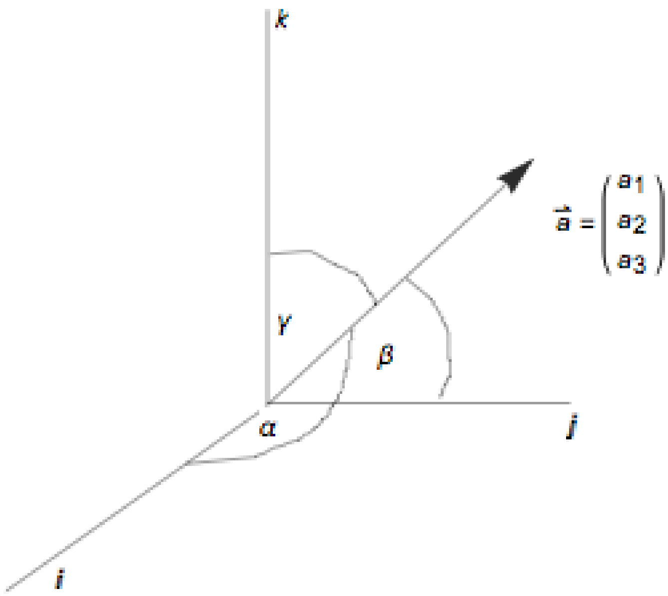
$$\frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|} \quad \text{and} \quad \frac{b_1}{|\vec{b}|}, \frac{b_2}{|\vec{b}|}, \frac{b_3}{|\vec{b}|} \tag{3}$$

One of the basic facts about direction cosines is that if θ is the angle between $\vec{a}$ and $\vec{b}$ the following relationship (in 3 dimensions) holds:

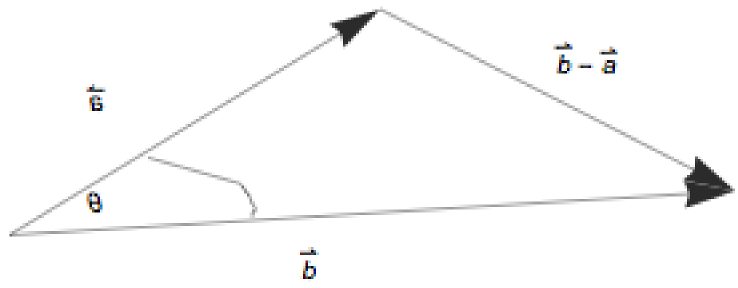
$$\cos\theta = \frac{a_1}{|\vec{a}|}\frac{b_1}{|\vec{b}|} + \frac{a_2}{|\vec{a}|}\frac{b_2}{|\vec{b}|} + \frac{a_3}{|\vec{a}|}\frac{b_3}{|\vec{b}|} \tag{4}$$

Equation (4) then gives the required equivalence after multiplying through by $|\vec{a}|\,|\vec{b}|$. The concept of direction cosines is set out in the figure below. By dropping the perpendiculars to each of the coordinate axes we see that:

$$\cos\alpha = \frac{a_1}{|\vec{a}|} \qquad \cos\beta = \frac{a_2}{|\vec{a}|} \qquad \cos\gamma = \frac{a_3}{|\vec{a}|} \tag{5}$$



We can also use the cosine law to derive (4).



Thus we have:

$$
\begin{aligned}
|\vec{b} - \vec{a}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}|\,\cos\theta \\
(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 &= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2|\vec{a}|\,|\vec{b}|\,\cos\theta \\
a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2(a_1 b_1 + a_2 b_2 + a_3 b_3) &= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2|\vec{a}|\,|\vec{b}|\,\cos\theta \\
\therefore \quad a_1 b_1 + a_2 b_2 + a_3 b_3 &= |\vec{a}|\,|\vec{b}|\,\cos\theta
\end{aligned}
\tag{6}
$$

In two dimensions one can also demonstrate the equivalence as follows (see the following diagram). Recall that $\cos(x-y) = \cos x \cos y + \sin x \sin y$ and $\cos(\cos^{-1} x)) = x$:

$\theta = \beta - \alpha$

$\cos\alpha = \frac{b_1}{|\vec{b}|} \implies \alpha = \cos^{-1}\left(\frac{b_1}{|\vec{b}|}\right)$

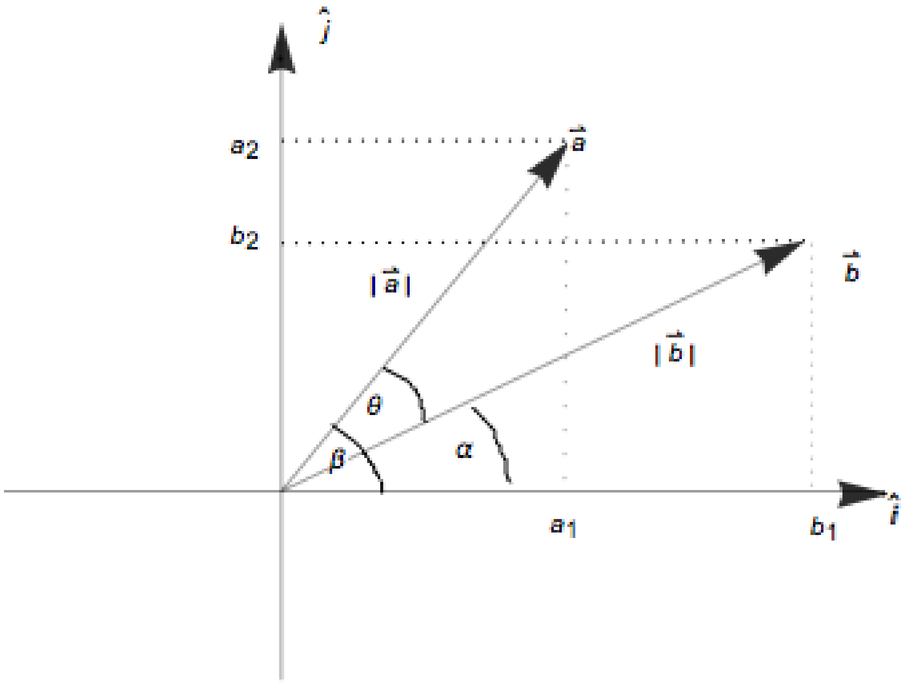
$\cos\beta = \frac{a_1}{|\vec{a}|} \implies \beta = \cos^{-1}\left(\frac{a_1}{|\vec{a}|}\right)$

$|\vec{a}|\,|\vec{b}|\,\cos\theta = |\vec{a}|\,|\vec{b}|\,\cos(\beta - \alpha) = |\vec{a}|\,|\vec{b}|\,\cos\left(\cos^{-1}\left(\frac{a_1}{|\vec{a}|}\right) - \cos^{-1}\left(\frac{b_1}{|\vec{b}|}\right)\right)$

$= |\vec{a}|\,|\vec{b}|\left(\frac{a_1}{|\vec{a}|}\,\frac{b_1}{|\vec{b}|} + \sin\alpha\,\sin\beta\right)$

$= |\vec{a}|\,|\vec{b}|\left(\frac{a_1}{|\vec{a}|}\,\frac{b_1}{|\vec{b}|} + \frac{b_2}{|\vec{b}|}\,\frac{a_2}{|\vec{a}|}\right)$

$= a_1 b_1 + a_2 b_2$



We could equally derive the relationship as follows. Let $\hat{a}$ and $\hat{b}$ be unit vectors in the directions of $\vec{a}$ and $\vec{b}$ respectively. We then write $\hat{a}$ and $\hat{b}$ in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$:

$$\begin{aligned}\hat{a} &= \cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}\\ \hat{b} &= \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}\end{aligned} \tag{7}$$

Now we take the dot product of $\hat{a}$ and $\hat{b}$ and recall that $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = 1$ and $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{\imath} = 0$. Note that:

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} \qquad \text{and } \hat{b} = \frac{\vec{b}}{|\vec{b}|} \tag{8}$$

$$\begin{aligned}\hat{a}\cdot\hat{b} &= (\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath})\cdot(\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath})\\ &= \cos\beta\,\cos\alpha\,\hat{\imath}\cdot\hat{\imath} + \cos\beta\,\sin\alpha\,\hat{\imath}\cdot\hat{\jmath} + \sin\beta\,\cos\alpha\,\hat{\jmath}\cdot\hat{\imath} + \sin\beta\,\sin\alpha\,\hat{\jmath}\cdot\hat{\jmath}\\ &= \cos\beta\,\cos\alpha + \sin\beta\,\sin\alpha\\ &= \cos(\beta - \alpha)\\ &= \cos\theta\end{aligned} \tag{9}$$

Therefore using (8):

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta \tag{10}$$

However since $\hat{a} \cdot \hat{b} = \cos\beta\, \cos\alpha + \sin\beta\, \sin\alpha$ we simply substitute the corresponding ratios as follows:

$$\begin{aligned} \hat{a} \cdot \hat{b} &= \cos\beta\, \cos\alpha + \sin\beta\, \sin\alpha \\ &= \frac{a_1}{|\vec{a}|} \frac{b_1}{|\vec{b}|} + \frac{a_2}{|\vec{a}|} \frac{b_2}{|\vec{b}|} \end{aligned} \tag{11}$$

Therefore:

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \Big(\frac{a_1}{|\vec{a}|} \frac{b_1}{|\vec{b}|} + \frac{a_2}{|\vec{a}|} \frac{b_2}{|\vec{b}|} \Big) = a_1 b_1 + a_2 b_2 \tag{12}$$

The concept of the dot product is generalised in differential geometry and tensor analysis. For instance, the angle γ between two vectors $\vec{a}$ and $\vec{b}$ is:

$$\cos\gamma = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{g_{\alpha\beta}\, a^\alpha b^\beta}{\sqrt{g_{\mu\nu} a^\mu a^\nu}\, \sqrt{g_{\sigma\,\tau} b^\sigma b^\tau}} \tag{13}$$

See Erwin Kreyszig, Differential Geometry , Dover, 1991, page 110.

1 History

Created 19 October 2014
09/09/2016 - corrected two typos on page 4 ie $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = 0$ and $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{\imath} = 1$ was an unfortunate transposition picked up by a reader