Dirichlet's test for convergence

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1 Statement of the test

Dirichlet's test for convergence is formulated this way by Elias Stein and Rami Shakarchi in "Fourier Analysis: An Introduction", Princeton Lectures in Analysis, Princeton University Press, 2003 at page 60 (Exercise 6):

Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$. If the partial sums of the series $\sum b_n$ are bounded and $\{a_n\}$ is a sequence of real numbers that decreases monotonically to zero, then $\sum a_n b_n$ converges.

The proof of this test is Exercise 6 in Stein and Skakarchi's book and involves a preliminary step which is the proof of the summation by parts formula:

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n \tag{1}$$

2 Roadmap for the proof

Because Dirichlet's test is asserting the convergence of $\sum a_n b_n$ without knowing the limit it must utilise the Cauchy criterion. Once you appreciate that point, the summation by parts formula in (1) is the essence of the proof. The Cauchy criterion applies to the LHS of (1) and to make it small (in absolute value) the absolute value of the RHS (after using the triangle inequality) can be made small because of the boundedness of the B_k

and the monotonic characteristics of the a_k. In what follows I flesh out the details. I ignore the complex nature of the sequences since any estimates involve absolute values of the real and imaginary parts ie real numbers.

2.1 Proof of the summation by parts formula

To prove (1), start with a low order case or two to get the structure as follows:

$$a_1b_1+a_2b_2 = a_1b_1+a_2(b_1+b_2)-a_2b_1 = (a_1-a_2)B_1+a_2B_2 = a_2B_2-\sum_{k=1}^{1}(a_{k+1}-a_k)B_k \tag{2}$$

Doing the same for n=3 we get:

$$\begin{aligned} a_1b_1 + a_2b_2 + a_3b_3 &= a_1b_1 + a_2(b_1 + b_2) - a_2b_1 + a_3(b_1 + b_2 + b_3) - a_3b_1 - a_3b_2 \\ &= (a_1 - a_2)B_1 + (a_2 - a_3)B_2 + a_3B_3 = a_3B_3 - \sum_{k=1}^{2}(a_{k+1} - a_k)B_k \end{aligned} \tag{3}$$

All we do now is prove inductively that:

$$\sum_{k=1}^{n} a_kb_k = a_nB_n - \sum_{k=1}^{n-1}(a_{k+1} - a_k)B_k \tag{4}$$

and then use the result to establish (1).

We have already established that the formula in (4) is true for n=2 (as well as n=3). Assume it is true for any n. Then:

$$\begin{aligned} \sum_{k=1}^{n+1} a_kb_k = \sum_{k=1}^{n} a_kb_k + a_{n+1}b_{n+1} &= a_nB_n - \sum_{k=1}^{n-1}(a_{k+1} - a_k)B_k + a_{n+1}b_{n+1} \\ &= a_nB_n - \sum_{k=1}^{n}(a_{k+1} - a_k)B_k + (a_{n+1} - a_n)B_n + a_{n+1}b_{n+1} \\ &= a_{n+1}b_{n+1} + a_{n+1}B_n - \sum_{k=1}^{n}(a_{k+1} - a_k)B_k = a_{n+1}B_{n+1} - \sum_{k=1}^{n}(a_{k+1} - a_k)B_k \end{aligned} \tag{5}$$

Thus (4) is true for all $n \geq 2$. To prove (1) we simply perform a substraction as follows:

$$
\begin{aligned}
\sum_{k=M}^{N} a_k b_k &= \sum_{k=1}^{N} a_k b_k - \sum_{k=M}^{M-1} a_k b_k \\
&= a_N B_N - \sum_{k=1}^{N-1} (a_{k+1} - a_k) B_k - \Big\{ a_{M-1} B_{M-1} - \sum_{k=1}^{M-2} (a_{k+1} - a_k) B_k \Big\} \\
&= a_N B_N - a_{M-1} B_{M-1} - \Big\{ \sum_{k=1}^{N-1} (a_{k+1} - a_k) B_k - \sum_{k=1}^{M-2} (a_{k+1} - a_k) B_k \Big\} \\
&= a_N B_N - a_{M-1} B_{M-1} - \sum_{k=M-1}^{N-1} (a_{k+1} - a_k) B_k \\
&= a_N B_N - a_{M-1} B_{M-1} - \sum_{k=M}^{N-1} (a_{k+1} - a_k) B_k - (a_M - a_{M-1}) B_{M-1} \\
&= a_N B_N - a_M B_{M-1} - \sum_{k=M}^{N-1} (a_{k+1} - a_k) B_k \qquad (6)
\end{aligned}
$$

2.2 Proving the Cauchy property

If we can prove that the partial sums $S_n = \sum_{k=1}^{n} a_k b_k$ satisfy the Cauchy criterion then it follows that the series $\sum a_k b_k$ converges. We know that the $B_k = \sum_{j=1}^{k} b_k$ are bounded ie there exists a $B > 0$ such that $|B_k| \leq B$ for all k. Because the $\{a_n\}$ monotonically decrease to 0 we know that $a_n \to 0$ and that $\{a_n\}$ therefore satisfies the Cauchy criterion. Thus we know that we can an M' such that $|a_n - a_m| < \epsilon$ for all $n > m > M'$. Because $a_n \to 0$ we can find a large enough M' such that $|a_n| < \epsilon$ for all $n > M'$. We can adjust these estimates to fit in with the inequalities set out below.

To see if the partial sums satisfy the Cauchy criterion we proceed as follows:

$$
\begin{aligned}
|S_N - S_{M-1}| = \Big| \sum_{k=1}^{N} a_k b_k - \sum_{k=1}^{M-1} a_k b_k \Big| = \Big| \sum_{k=M}^{N} a_k b_k \Big| &= \Big| a_N B_N - a_M B_{M-1} - \sum_{k=M}^{N-1} (a_{k+1} - a_k) B_k \Big| \\
&\leq \big| a_N B_N - a_M B_{M-1} \big| + \sum_{k=M}^{N-1} \big| (a_{k+1} - a_k) \big| \, \big| B_k \big| \qquad (7)
\end{aligned}
$$

By choosing M' sufficiently large we can ensure that we can make $|a_N - a_M| < \frac{\epsilon}{3B}$ for all $N > M > M'$. The final term in (7) is therefore at most $(N-M)B\frac{\epsilon}{3B}$ which is less than $\frac{NB\epsilon}{3B} = \frac{N\epsilon}{3}$. Hence if we choose M' so large that for $N > M > M'$, $|a_N - a_M| < \frac{\epsilon}{3BN}$

then the final term will be be less than $\frac{\epsilon}{3}$.

For the first part of (7):

$\left|a_N B_N - a_M B_{M-1}\right| \leq |a_N|B + |a_M|B < \frac{\epsilon}{3B}B + \frac{\epsilon}{3B}B = \frac{2\epsilon}{3}$ because we can choose an M' large enough so that both $|a_N|$ and $|a_M|$ can be made less than $\frac{\epsilon}{3B}$ for all $N > M > M'$.

Thus if we choose this M' to be sufficiently large we can ensure that all three component estimates within (7) can be in total less than ϵ which means that the sequence of partial sums is Cauchy and the series $\sum a_k b_k$ converges.