- **HSC integral reduction formulas**

While there is a relatively limited suite of integral reduction formulas that the examiners can throw at HSC students, this does not mean that they are necessarily easy to nail in the heat of an exam. The following examples may assist in solving this style of problem.

- **If $I_n = \int_0^1 x\,(1 - x^3)^n\,dx$ for n ≥ 0 show that $I_n = \frac{3n}{3n+2}\, I_{n-2}$ for n ≥ 1 and find an expression for I_n in terms of n for n ≥ 0.**

$I_n = \int_0^1 x\,(1 - x^3)^n dx$

$= \int_0^1 x\,(1 - x^3)^{n-1}\,(1 - x^3)\,dx$

$= \int_0^1 x\,(1 - x^3)^{n-1}\,dx - \int_0^1 x^4\,(1 - x^3)^{n-1}\,dx$

$= I_{n-1} - J$ (1)

Now I_n can be integrated by parts as follows:

$I_n = \int_0^1 x\,(1 - x^3)^n dx$ where dv = x dx and u = $(1 - x^3)^n$ so that $v = \frac{x^2}{2}$ and du = $-3nx^2\,(1 - x^3)^{n-1}dx$

Hence $I_n = \frac{x^2}{2}\,(1 - x^3)^n \Big|_0^1 + \frac{3n}{2}\int_0^1 x^4\,(1 - x^3)^{n-1}dx$

$I_n = 0 + \frac{3n}{2} J$ (2)

Now we solve (1) and (2) simultaneously to find that:

$\frac{3n}{2} I_n = \frac{3n}{2} I_{n-1} - \frac{3n}{2} J$ (3)

(2) + (3): $(1 + \frac{3n}{2})\, I_n = \frac{3n}{2} I_{n-1}$

Therefore $I_n = \frac{3n}{3n+2} I_{n-1}$

We now proceed inductively (iteratively) as follows:

$I_n = \frac{3n}{3n+2} I_{n-1}$

$= \frac{3n}{3n+2}\,\frac{3n-3}{3n-1}\, I_{n-2}$

$= \frac{3n}{3n+2}\,\frac{3n-3}{3n-1}\,\frac{3n-6}{3n-4}\, I_{n-3}$

$= \frac{3^3\, n\,(n-1)\,(n-2)}{(3n+2)\,(3n-1)\,(3n-4)}\, I_{n-3}$

......

$= \frac{3^n\, n\,(n-1)\,(n-2)\ldots 2.1}{(3n+2)\,(3n-1)\,(3n-4)\ldots 8.5}\, I_0$ This is the critical step. For each reduction of n by 1 in the numerator (note that n! = n (n-1) (n-2)...2.1. has n terms) the denominator terms drops by 3. Thus the index starts at 3n+2 and to get from I_{n-1} to I_0 there are n-1 lots of -3. So the final term in the denominator is 3n +2 - 3(n-1) = 5.

Finally $I_0 = \int_0^1 x\,dx = \frac{1}{2}$ so that the formula is:

$I_n = \dfrac{3^n\, n!}{(3n+2)\,(3n-1)\,(3n-4)\ldots 8.5\ \ .2}$

As a quick check we note that $I_1 = \int_0^1 x\,(1-x^3)\,dx = [\,\frac{x^2}{2} - \frac{x^5}{5}\,]_0^1 = \frac{3}{10}$ while the formula gives:

$I_1 = \frac{3}{5} I_0 = \frac{3}{10}$

Another angle is that you could be given the formula and then asked to prove it by a formal inductive argument.

- **If $I_n = \int_0^a (a^2 - x^2)^n\, dx$ for $n \geq 0$ show that $I_n = \frac{2na^2}{2n+1} I_{n-1}$ for $n \geq 2$ and find I_3**

In $I_n = \int_0^a (a^2 - x^2)^n dx$ let dv = dx so that v = x and u = $(a^2 - x^2)^n$ so that du = $-2nx\,(a^2 - x^2)^{n-1}dx$

Hence $I_n = x(a^2 - x^2)^n \Big|_0^a + 2n \int_0^a x^2\,(a^2 - x^2)^{n-1}dx$

$= 2n \int_0^a x^2\,(a^2 - x^2)^{n-1}dx$

$I_n = 2n\,J$ (1)

Also $I_n = \int_0^a (a^2 - x^2)^{n-1}\,(a^2 - x^2)\,dx$

$= a^2 \int_0^a (a^2 - x^2)^{n-1}\,dx - \int_0^a x^2\,(a^2 - x^2)^{n-1}\,dx$

$= a^2 I_{n-1} - J$ (2)

Now I_n can be integrated by parts as follows:

We solve (1) and (2) simultaneously to find that:

$I_n = a^2 I_{n-1} - \frac{I_n}{2n}$

$(1 + \frac{1}{2n})\,I_n = a^2 I_{n-1}$

Therefore $I_n = \frac{2na^2}{2n+1} I_{n-1}$

We now proceed inductively (iteratively) as follows:

$I_n = \frac{2na^2}{2n+1} I_{n-1}$

$= \frac{2na^2}{2n+1}\,\frac{2(n-1)a^2}{2n-1}\,I_{n-2}$

$= \frac{2na^2}{2n+1}\,\frac{2(n-1)a^2}{2n-1}\,\frac{2(n-2)a^2}{2n-3}\,I_{n-3}$

......

$= \frac{2na^2}{2n+1}\,\frac{2(n-1)a^2}{2n-1}\,\frac{2(n-2)a^2}{2n-3}\,..\,\frac{2\times 2a^2}{5}\,I_1$

$= \frac{2na^2}{2n+1}\,\frac{2(n-1)a^2}{2n-1}\,\frac{2(n-2)a^2}{2n-3}\,..\,\frac{2\times 2a^2}{5}\,\frac{2\times 1a^2}{3}\,I_0$

$= \frac{2na^2}{2n+1}\,\frac{2(n-1)a^2}{2n-1}\,\frac{2(n-2)a^2}{2n-3}\,..\,\frac{2\times 2a^2}{5}\,\frac{2\times 1a^2}{3}\,a$ since $I_0 = \int_0^a (a^2 - x^2)^0 dx = a$

$$= \frac{2^n\, n!\, a^{2n+1}}{(2n+1)(2n-1)(2n-3)\ldots 5.3}$$

As a quick check we note that $I_1 = \int_0^a (a^2 - x^2)^1 dx = [\, a^2 x - \frac{x^3}{3}]_0^a = \frac{2}{3} a^3$ while the formula gives:

$$I_1 = \frac{2a^2}{3} I_0 = \frac{2a^2}{3} a = \frac{2}{3} a^3$$

$$I_3 = \frac{2^3\, 3!\, a^7}{7.5 \quad .3} = \frac{16 a^7}{35}$$

$$I_3 = \int_0^a (a^2 - x^2)^3 dx = \int_0^a (a^6 - 3a^4 x^2 + 3a^2 x^4 - x^6) dx = [a^6 x - \frac{3a^4 x^3}{3} + \frac{3a^2 x^5}{5} - \frac{x^7}{7}]_0^a$$

$$= a^7 - a^7 + \frac{3a^7}{5} - \frac{a^7}{7}$$

$= \frac{16 a^7}{35}$ as per the formula.

- **If $u_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ for $n \geq 0$ show that $u_n + u_{n-2} = 1$ for $n \geq 2$ and hence evaulate the integral for all positive integral values of n**

$$u_n = \int_0^{\frac{\pi}{4}} (\tan^{n-2} x \tan^2 x \; dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan^{n-2} x (\sec^2 x - 1)\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \; dx - \int_0^{\frac{\pi}{4}} (\tan^{n-2} x \; dx$$

$= \int_0^1 v^{n-2}\, dv - u_{n-2}$ using the substitution $v = \tan x \Rightarrow dv = \sec^2 x \; dx$

$$= \frac{1}{n-1} - u_{n-2}$$

Therefore $u_n + u_{n-2} = \frac{1}{n-1}$

Note that $u_0 = \int_0^{\frac{\pi}{4}} dx = \frac{\pi}{4}$ and $u_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx = \frac{\ln 2}{2}$

We can develop formulas for n even and n odd as follows:

$$u_2 = 1 - u_0$$

$$u_4 = \frac{1}{3} - u_2 = \frac{1}{3} - 1 + u_0$$

$$u_6 = \frac{1}{5} - u_4 = \frac{1}{5} - \frac{1}{3} + 1 - u_0$$

$$u_8 = \frac{1}{7} - u_6 = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + u_0$$

.....

$$u_{2n} = (-1)^n \{ \sum_{k=1}^n \frac{(-1)^k}{2k-1} + \frac{\pi}{4} \}$$

Checking this formula for n= 3 we see that $u_6 = (-1)^3 \{ \sum_{k=1}^3 \frac{(-1)^k}{2k-1} + \frac{\pi}{4} \} = -\{-1 + \frac{1}{3} - \frac{1}{5} + \frac{\pi}{4}\}$

$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{\pi}{4}$ which is correct.

One can do an inductive proof of this formula and the critical inductive step is as follows (mechanical details omitted):

$$u_{2n+2} + u_{2n} = \frac{1}{2n+1}$$

$$u_{2n+2} = \frac{1}{2n+1} - u_{2n}$$

$$= \frac{1}{2n+1} - (-1)^n \left\{ \sum_{k=1}^{n} \frac{(-1)^k}{2k-1} + \frac{\pi}{4} \right\}$$

$$= \frac{1}{2n+1} + (-1)^{n+1} \left\{ \sum_{k=1}^{n} \frac{(-1)^k}{2k-1} + \frac{\pi}{4} \right\}$$

$$= (-1)^{n+1} \left\{ \sum_{k=1}^{n+1} \frac{(-1)^k}{2k-1} + \frac{\pi}{4} \right\}$$ so that the formula is true for n+1.

One can follow exactly the same logic for odd values to get the following formula:

$$u_{2n+1} = (-1)^n \left\{ \sum_{k=1}^{n} \frac{(-1)^k}{2k} + \frac{\ln 2}{2} \right\}$$

Thus $$\int_0^{\frac{\pi}{4}} \tan^n x \, dx = \begin{cases} (-1)^{n/2} \left\{ \sum_{k=1}^{n/2} \frac{(-1)^k}{2k-1} + \frac{\pi}{4} \right\} & n \text{ even} \\ (-1)^{\frac{n-1}{2}} \left\{ \sum_{k=1}^{\frac{n-1}{2}} \frac{(-1)^k}{2k} + \frac{\ln 2}{2} \right\} & n \text{ odd} \end{cases}$$

- **If $u_{m,n} = \int_0^1 x^m (1-x)^n \, dx$ for m,n integers ≥ 1 show that $(m+n+1)\, u_{m,n} = n\, u_{m,n-1}$ and deduce that $u_{m,n} = \frac{m!\, n!}{(m+n+1)!}$**

In $u_{m,n} = \int_0^1 x^m (1-x)^n \, dx$

$$= \int_0^1 x^m (1-x)^{n-1} (1-x) \, dx$$

$$= \int_0^1 x^m (1-x)^{n-1} \, dx - \int_0^1 x^{m+1} (1-x)^{n-1} \, dx$$

$$= u_{m,n-1} - u_{m+1,n-1} \qquad (1)$$

In $u_{m+1,n-1} = \int_0^1 x^{m+1} (1-x)^{n-1} \, dx$ let $u = x^{m+1}$ and $du = (m+1)\, x^m \, dx$ and $dv = (1-x)^{n-1} \, dx$ so that $v = \frac{-(1-x)^n}{n}$

Hence $$u_{m+1,n-1} = \left.\frac{-x^{m+1}}{n} (1-x)^n \right]_0^1 + \frac{m+1}{n} \int_0^1 x^m (1-x)^n \, dx$$

$$= \frac{m+1}{n} u_{m,n} \qquad (2)$$

Substituting (2) into (1) we have that:

$$u_{m,n} = u_{m,n-1} - \frac{m+1}{n} u_{m,n}$$

$$\frac{m+n+1}{n} u_{m,n} = u_{m,n-1}$$

So $(m+n+1)\ u_{m,n} = n\ u_{m,n-1}$ as claimed.

For the next step we treat m as fixed and vary n and we write $u_{m,n} = u_n$ for convenience with this assumption.

We proceed inductively:

$$u_n = \frac{n}{m+n+1} u_{n-1}$$

$$= \frac{n}{m+n+1} \frac{n-1}{m+n} u_{n-2}$$

$$= \frac{n}{m+n+1} \frac{n-1}{m+n} \frac{n-2}{m+n-1} u_{n-3}$$

....

$= \frac{n}{m+n+1} \frac{n-1}{m+n} \frac{n-2}{m+n-1} \ldots \frac{1}{m+2} u_0$ where $u_0 = \int_0^1 x^m\, dx = \frac{1}{m+1}$. Note that to get from u_{n-1} to u_0 there are n-1 steps so that the last term in the top is 1 and in the bottom it is m+n+1 - (n-1) = m+2.

$$= \frac{n}{m+n+1} \frac{n-1}{m+n} \frac{n-2}{m+n-1} \ldots \frac{1}{m+2} \frac{1}{m+1}$$

$$= \frac{n!\, m!}{(n+m+1)!}$$

Hence $u_{m,n} = \frac{n!\, m!}{(n+m+1)!}$ as advertised.

- **If $I_n = \int \sin^n x\, dx$ for integers n ≥ 0 then show that**
  $I_n = \frac{-1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$ for n ≥ 2 . Hence show that if
  $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$ for n ≥ 0 then $I_n = \frac{n-1}{n} I_{n-2}$ for n ≥ 2 and then deduce that $I_5 . I_6 = \frac{\pi}{12}$

In $I_n = \int \sin^n x\, dx = \int \sin^{n-1} x\ \sin x\, dx$ let dv = sin x dx so that v = - cos x and let $u = \sin^{n-1} x$ so that du = $(n-1)\sin^{n-2} x \cos x\, dx$

Hence $I_n = -\cos x \sin^{n-1} x + (n-1) \int \cos^2 x \sin^{n-2} x\, dx$

$$= -\cos x \sin^{n-1} x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x\, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

$$I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

Therefore $nI_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$

$$I_n = \frac{-1}{n} \cos x \sin^{n-1} x + \frac{(n-1)}{n} I_{n-2} \quad \text{for } n \geq 2 \quad ....(1)$$

Now when $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$ the first part of (1) collapses to zero at both x = 0 and $x = \frac{\pi}{2}$ hence

$$I_n = \frac{(n-1)}{n} I_{n-2} \quad \text{for for } n \geq 2$$

Now $I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$

$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_0^{\pi/2} = 1$

$I_2 = \frac{1}{2} I_0 = \frac{\pi}{4}$

$I_5 . I_6 = \frac{4}{5} I_3 \; \frac{5}{6} I_4 = \frac{4}{5} \frac{2}{3} I_1 \; \frac{5}{6}\frac{3}{4} I_2 = \frac{4}{5} \frac{2}{3} 1 \; \frac{5}{6}\frac{3}{4}\frac{\pi}{4} = \frac{\pi}{12}$

- **If $I_n = \int \sec^n x \, dx$ for integers $n \geq 0$ then show that $I_n = \frac{1}{n-1} \tan x \sec^{n-2} x + \frac{n-2}{n-1} I_{n-2}$ for $n \geq 2$. Hence show that if $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$ for $n \geq 0$ then $I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ for $n \geq 2$ and then deduce that $I_6 = \frac{28}{15}$**

In $I_n = \int \sec^n x \, dx = \int \sec^{n-2} x \; \sec^2 x \, dx$ let dv = $\sec^2 x \, dx$ so that v = tan x and let u = $\sec^{n-2} x$ so that du = (n-2)$\sec^{n-3} x$ sec x tan x dx = (n-2)$\sec^{n-2} x$ tan x dx

Hence $I_n = \tan x \sec^{n-2} x - (n-2) \int \tan^2 x \sec^{n-2} x \, dx$

$= \tan x \sec^{n-2} x - (n-2) \int (\sec^2 x - 1) \sec^{n-2} x \, dx$

$= \tan x \sec^{n-2} x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$

$I_n = \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$

Therefore $(n-1) I_n = \tan x \sec^{n-2} x + (n-2) I_{n-2}$

$I_n = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$ ....(1) for $n \geq 2$

Now when $I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$ the first part of (1) collapses to $\frac{(\sqrt{2})^{n-2}}{n-1}$ hence

$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ for for $n \geq 2$

Now $I_2 = \int_0^{\frac{\pi}{4}} \sec^2 x \, dx = [\tan x]_0^{\pi/4} = 1$

From (1):

$I_6 = \frac{4}{5} + \frac{4}{5} I_4$

$I_4 = \frac{2}{3} + \frac{2}{3} I_2 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

Hence $I_6 = \frac{4}{5} + \frac{4}{5} \frac{4}{3} = \frac{28}{15}$

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