Hermite functions - a solution to a Stein and Shakarchi problem

Peter Haggstrom
www.gotohaggstrom.com
mathsatbondibeach@gmail.com

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1 Introduction

In Eli Stein and Rami Shakarchi's Fourier Analysis textbook ([6], page 173] there is an extensive problem on Hermite functions (the pronunciation of 'Hermite' is given here: `https://www.gotohaggstrom.com/Hermite.mov`). Because Stein's emphasis is on analysis, the development of the exposition is different to that one might find in a quantum physics context for instance (see [2] pages 529-534 for the development in the context of the harmonic oscillator). The problem has several parts and I will start by setting out the problem as posed. It is an asterisked problem so it is harder than other problems or it goes further in terms of the relevant theory. Indeed, it builds upon two previous exercises/problems which themselves require a deal of work so this is a really "meaty" problem.

***Problem* 7** ([6], page 173)

The Hermite functions $h_k(x)$ are defined by the generating identity:

$$\sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!} = e^{-(\frac{x^2}{2}-2tx+t^2)} \tag{1}$$

(a) Show that an alternate definition of the Hermite functions is given by the formula:

$$h_k(x) = (-1)^k e^{\frac{x^2}{2}}\left(\frac{d}{dx}\right)^k e^{-x^2} \tag{2}$$

[Hint: Write $e^{-(\frac{x^2}{2}-2tx+t^2)} = e^{\frac{x^2}{2}}e^{-(x-t)^2}$ and use Taylor's formula.] Conclude from the above expression that each $h_k(x)$ is of the form $P_k(x)e^{-\frac{x^2}{2}}$, where $P_k(x)$ is a polynomial of degree k. In particular, the Hermite functions belong to the Schwartz space and $h_0(x) = e^{\frac{-x^2}{2}}$, $h_1(x) = 2xe^{\frac{-x^2}{2}}$.

(b) Prove that $\{h_k\}_{k=0}^{\infty}$ is complete in the sense that if f is a Schwartz function and:

$$(f, h_k) = \int_{-\infty}^{\infty} f(x) h_k(x)\, dx = 0 \quad \text{for all } k \geq 0 \tag{3}$$

then $f = 0$. [Hint: Use Exercise 8]

***Exercise* 8** ([6], page 163) is as follows:

Prove that if f is continuous, of moderate decrease, and $\int_{-\infty}^{\infty} f(y) e^{-y^2} e^{2xy}\, dy = 0$ for all $x \in \mathbb{R}$ then $f = 0$. [Hint: Consider $f * e^{-x^2}$.]

Note that a function f defined on $\mathbb{R}$ is said to be of moderate decrease if f is continuous and there exists a constant $A > 0$ such that:

$$|f(x)| \leq \frac{A}{1 + x^2} \quad \text{for all } x \in \mathbb{R} \tag{4}$$

(c) Define $h_k^*(x) = h_k\big(\sqrt{2\pi} x\big)$. Then:

$$\widehat{h_k^*}(\xi) = (-i)^k h_k^*(\xi) \tag{5}$$

Therefore, each h_k^* is an eigenfunction for the Fourier transform.

(d) Show that h_k is an operator for the eigenfunction for the operator defined in Exercise 23 (see below) and, in fact, prove that:

$$L h_k = (2k + 1) h_k \tag{6}$$

In particular, we conclude that the functions h_k are mutually orthogonal for the $\mathcal{L}^2$ inner product on the Schwartz space.

(e) Finally, show that $\int_{-\infty}^{\infty} |h_k(x)|^2\, dx = \sqrt{\pi}\, 2^k k!$. [Hint: Square the generating relation].

***Exercise* 23** ([6], pages 168-169) :

The Heisenberg uncertainty principle can be formulated in terms of the operator $L = -\frac{d^2}{dx^2} + x^2$ which acts on Schwartz functions by the formula:

$$L(f) = -\frac{d^2 f}{dx^2} + x^2 f \tag{7}$$

This operator, sometimes called the Hermite operator, is the quantum analogue of the harmonic oscillator, Consider the usual inner product on the Schwartz space $\mathcal{S}$ given by:

$$(f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)}\, dx \quad \text{whenever f,g} \in \mathcal{S} \tag{8}$$

(a) Prove that the Heisenberg uncertainty principle implies:

$$(Lf, f) \geq (f, f) \quad \text{for all f} \in \mathcal{S} \tag{9}$$

This is usually denoted by $L \geq I$. [Hint: Integrate by parts].

(b) Consider the operators A and A^* defined on $\mathcal{S}$ by:

$$A(f) = \frac{df}{dx} + xf \tag{10}$$

$$A^*(f) = -\frac{df}{dx} + xf \tag{11}$$

The operators A and A^* are sometimes called the annihilation and creation operators respectively. Prove that for all f, g $\in \mathcal{S}$ we have:

(i)

$$(Af, g) = (f, A^*g) \tag{12}$$

(ii)

$$(Af, Af) = (A^*Af, f) \geq 0 \tag{13}$$

(iii)

$$A^*A = L - I \tag{14}$$

In particular this again shows that $L \geq I$.

(c) Now for $t \in \mathbb{R}$ let:

$$A_t(f) = \frac{df}{dx} + txf \tag{15}$$

$$A_t^*(f) = -\frac{df}{dx} + txf \tag{16}$$

Use the fact that $(A_t^* A_t f, f) \geq 0$ to give another proof of the Heisenberg uncertainty principle which says that whenever $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = 1$ then:

$$\left(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx \right) \left(\int_{-\infty}^{\infty} \left| \frac{df}{dx} \right|^2 dx \right) \geq \frac{1}{4} \tag{17}$$

[Hint: Think of $(A_t^* A_t f, f)$ as a quadratic polynomial in t.]

2 Solution

2.1 Problem 7 Part (a)

(a) We have:

$$\begin{aligned}
e^{-(\frac{x^2}{2}-2tx+t^2)} &= e^{\frac{x^2}{2}}\, e^{-(x-t)^2} \\
&= e^{\frac{x^2}{2}}\left[e^{-x^2} + \frac{(-t)}{1!}\frac{d}{dx}\left(e^{-x^2}\right) + \frac{(-t)^2}{2!}\frac{d^2}{dx^2}\left(e^{-x^2}\right) + \dots\right] \\
&= e^{\frac{x^2}{2}}\sum_{k=0}^{\infty}(-1)^k\frac{d^k}{dx^k}\left(e^{-x^2}\right)\frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} h_k(x)\,\frac{t^k}{k!} \quad \text{with } h_k(x) = (-1)^k e^{\frac{x^2}{2}}\frac{d^k}{dx^k}\left(e^{-x^2}\right)
\end{aligned} \tag{18}$$

Note here that Taylor's theorem has the form $f(x-t) = f(x) - \frac{t}{1!}\frac{df}{dx} + \frac{t^2}{2!}\frac{d^2f}{dx^2} + \dots$ with $f(x) = e^{-x^2}$.

That $h_k(x)$ is of the form $P_k(x)e^{-\frac{x^2}{2}}$ follows inductively. We start trivially with $e^{\frac{x^2}{2}}\frac{d^0}{dx^0}\left(e^{-x^2}\right) = e^{\frac{x^2}{2}}e^{-x^2} = e^{-\frac{x^2}{2}}$ so that $h_0(x) = e^{-\frac{x^2}{2}}$ and so is the product of zero degree polynomial (ie a constant) and $e^{-\frac{x^2}{2}}$. Slightly more interestingly with $k=1$ we have $\frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}$ and hence $h_1(x) = -e^{\frac{x^2}{2}} \times -2xe^{-x^2} = P_1(x)e^{-\frac{x^2}{2}}$ where $P_1(x)$ is a polynomial of degree 1.

Using the obvious induction hypothesis we also have:

$$\frac{d^{k+1}}{dx^{k+1}}\left(e^{-x^2}\right) = \frac{d}{dx}\left(P_k(x)e^{-x^2}\right) = [\quad \underbrace{-2xP_k(x)}_{\text{poly of degree } k+1} \quad + \quad \underbrace{P_k'(x)}_{\text{poly of degree } k-1} \quad]\, e^{-x^2} \tag{19}$$

Hence $\frac{d^{k+1}}{dx^{k+1}}\left(e^{-x^2}\right) = P_{k+1}(x)e^{-x^2}$ and the result follows. The Hermite functions inhabit Schwartz space because they are a product of a degree k polynomial and $e^{-\frac{x^2}{2}}$ and the exponential term decays faster than any polynomial. They are also differentiable an arbitrary number of times. To establish this in more detail we need a definition of what a Schwartz function is (see [3] pages,5-6). Thus the h_k will inhabit Schwartz space if:

$$\sup_{x\in\mathbb{R}} |x|^m \left|h_k^{(l)}(x)\right| < \infty \quad \forall m, l \geq 0 \tag{20}$$

The m, l can be chosen independently which is a very powerful requirement. The l^{th} derivative of $h_k(x)$ is the product of a $k+l$ degree polynomial and $e^{-\frac{x^2}{2}}$. This can be seen as follows (we use $P_k(x)$ to be any polynomial of degree k to avoid using multiple symbols for sums of polynomials of different degrees). Again it is an inductive style of approach. Using $D^l = \frac{d^l}{dx^l}$:

$$
\begin{aligned}
D(P_k(x)e^{-\frac{x^2}{2}}) =& e^{-\frac{x^2}{2}}\left[P_{k-1}(x) - xP_k(x)\right] \\
=& P_{k+1}(x)\, e^{-\frac{x^2}{2}} \\
D^2(P_k(x)e^{-\frac{x^2}{2}}) =& D(P_{k+1}(x)\, e^{-\frac{x^2}{2}}) \\
=& e^{-\frac{x^2}{2}}\left[P_k(x) - xP_{k+1}(x)\right] \\
=& P_{k+2}(x)e^{-\frac{x^2}{2}} \\
& \vdots \\
D^l(P_k(x)e^{-\frac{x^2}{2}}) =& P_{k+l}(x)e^{-\frac{x^2}{2}}
\end{aligned}
\tag{21}
$$

Thus given any $m,\, l \geq 0, |x|^m \,|P_{k+l}(x)|\, e^{-\frac{x^2}{2}}$ is bounded on $\mathbb{R}$ since the exponential dominates any polynomial because e^y tends to infinity with y faster than any power of y (for a short proof see [4], pages 407-8 which is esssentially reproduced in the Appendix).

Thus Hermite functions do inhabit Schwartz space.

2.2 Problem 7, Part b

(b) For this part we have to use a property established in Exercise 8, namely, that if f is continuous, of moderate decrease, and $\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}\,dy = 0$ for all $x \in \mathbb{R}$ then $f = 0$. [Hint: Consider $f * e^{-x^2}$.] To prove this let $g(x) = e^{-x^2}$. Then:

$$
(f * g)(x) = \int_{\infty}^{\infty} f(y)e^{-(x-y)^2}\,dy = e^{-x^2}\int_{\infty}^{\infty} f(y)e^{-y^2}e^{2xy}\,dy = 0 \quad \forall x \in \mathbb{R} \tag{22}
$$

Now we know that the Fourier transform of the convolution of f and g is the product of the Fourier transforms of f and g ie;

$$
\begin{aligned}
\widehat{f * g}(\xi) =& \hat{f}(\xi)\,\hat{g}(\xi) \\
=& \hat{f}(\xi)\,\sqrt{\pi}e^{-\pi^2\xi^2} \\
=& 0 \quad \forall \xi
\end{aligned}
\tag{23}
$$

Thus $\hat{f}(\xi) = 0$ and by the Fourier Inversion Theorem (see Theorem 1.9 page 141 of [6]) we have:

$$
f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}\,d\xi = 0 \quad \forall x \in \mathbb{R} \tag{24}
$$

Recall Corolllary 1.5 page 139 of [5] states that if $\delta > 0$ and $K_\delta(x) = \delta^{-\frac{1}{2}}e^{-\frac{\pi x^2}{\delta}}$, then $\hat{K}_\delta(\xi) = e^{-\pi\delta\xi^2}$. Hence $g(x) = \left(\frac{1}{\sqrt{\pi}}e^{-\frac{\pi x^2}{\pi}}\right)\sqrt{\pi}$ (ie $\delta = \pi$) and so $\hat{g}(\xi) = \sqrt{\pi}e^{-\pi^2\xi^2}$. Also note that since we have shown that the Hermite functions inhabit Schwartz space they are certainly of moderate decrease for the purposes of this exercise.

So now we are in a position to prove that if f is a Schwartz function and if:

$$(f, h_k) = \int_{-\infty}^{\infty} f(x) h_k(x)\, dx = 0 \quad \text{for all } k \geq 0 \tag{25}$$

then $f = 0$.

All we need to do is to write (25) as follows:

$$\begin{aligned}(f, h_k) = \int_{-\infty}^{\infty} f(x) h_k(x)\, dx = & \int_{-\infty}^{\infty} f(x) e^{-(\frac{x^2}{2} - 2tx + t^2)}\, dx \\ = & e^{-t^2} \int_{-\infty}^{\infty} f(x) e^{-\frac{x^2}{2}}\, e^{2tx}\, dx \\ = & 0\end{aligned} \tag{26}$$

Hence $f = 0$ using Exercise 8.

2.3 Problem 7, Part c

(c) For this part of the problem we assume that $h_k^*(x) = h_k\left(\sqrt{2\pi}x\right)$ and have to show that:

$$\widehat{h_k^*}(\xi) = (-i)^k h_k^*(\xi) \tag{27}$$

Using (1) we have:

$$\begin{aligned}\sum_{k=0}^{\infty} h_k^*(x) \frac{t^k}{k!} = \sum_{k=0}^{\infty} h_k(\sqrt{2\pi}x) \frac{t^k}{k!} = & e^{-(\pi x^2 - 2t\sqrt{2\pi}x + t^2)} \\ = & e^{-\pi\left(x^2 - \frac{2}{\pi} t\sqrt{2\pi}x + \frac{t^2}{\pi}\right)} \\ = & e^{-\pi\left[(x - t\sqrt{\frac{2}{\pi}})^2 + \frac{t^2}{\pi} - \frac{2t^2}{\pi}\right]} \\ = & e^{t^2} e^{-\pi[(x - t\sqrt{\frac{2}{\pi}})^2]}\end{aligned} \tag{28}$$

We now take Fourier transforms of both sides of (28) and in what follows we let $g(x) = e^{-\pi(x - t\sqrt{\frac{2}{\pi}})^2}$. Note also that the Fourier transform of a translation is as follows:

$$f(x+h) \xrightarrow{\mathcal{F}} \hat{f}(\xi) e^{2\pi i h \xi} \tag{29}$$

In this case $h = -t\sqrt{\frac{2}{\pi}}$ and $f(x) = e^{-x^2}$. Thus:

$$\hat{g}(\xi) = e^{-2\pi i t \sqrt{\frac{2}{\pi}}\xi}\, e^{-\pi\xi^2} = e^{-\pi\xi^2} e^{-2\sqrt{2\pi} i t \xi} \tag{30}$$

So the Fourier transforms of (28) become (noting that the factor e^{t^2} is viewed as a constant as we are taking transfroms with respect to x):

$$\begin{aligned}\sum_{k=0}^{\infty}\widehat{h_k^*}(\xi)\frac{t^k}{k!} =& e^{t^2}\, e^{-\pi\xi^2}\, e^{-2\sqrt{2\pi}it\xi} \\ =& e^{-\left(\pi\xi^2-2(-it)\sqrt{2\pi}\xi+(-it)^2\right)} \\ =& \sum_{k=0}^{\infty} h_k^*(\xi)\frac{(-it)^k}{k!}\end{aligned} \tag{31}$$

Therefore equating coefficients in (31) we have that $\widehat{h_k^*}(\xi) = (-i)^k h_k^*(\xi)$ as required so each h_k^* is indeed an eigenfunction of the Fourier transform.

2.4 Problem 7, Part d -introduction

(d) The solution to this problem revolves around Exercise 23 part (a) of which requires proving that the Heisenberg uncertainty principle implies (9). Stein explains the Heisenberg uncertainty principle in Theorem 4.1 page 158 of [6]. It reads as follows;

If $\Psi \in \mathcal{S}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} |\Psi(x)|^2\, dx = 1$ then:

$$\left(\int_{-\infty}^{\infty} x^2\, |\Psi(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} \xi^2 |\widehat{\Psi}(\xi)|^2\, d\xi\right) \geq \frac{1}{16\pi^2} \tag{32}$$

and equality holds if and only if $\Psi(x) = Ae^{-Bx^2}$ where $B > 0$ and $A^2 = \sqrt{\frac{2B}{\pi}}$.

2.5 Exercise 23 part (a)

Recalling the definition of the operator L in (7) we have:

$$\begin{aligned}(Lf, f) =& \int_{-\infty}^{\infty}\Big(-\frac{d^2 f(x)}{dx^2} + x^2 f(x)\Big)\,\overline{f(x)}\, dx \\ =& \int_{-\infty}^{\infty} -\frac{d^2 f(x)}{dx^2}\,\overline{f(x)}\, dx + \int_{-\infty}^{\infty} x^2 f(x)\,\overline{f(x)}\, dx \\ =& -(f^{''}, f) + (x^2 f, f)\end{aligned} \tag{33}$$

Without any loss of generality we can assume in what follows that $(f, f) = 1$ since we can divide by $\sqrt{(f, f)}$. We integrate the first term on the right hand side of (33) (ie $-(f^{''}, f)$) by parts as follows:

$$
\begin{aligned}
\int_{-\infty}^{\infty} -\frac{d^2 f(x)}{dx^2}\,\overline{f(x)}\,dx &= -\int_{-\infty}^{\infty} \overline{f(x)}\,d\Big(\frac{df(x)}{dx}\Big) \\
&= \underbrace{\Big[f^{'}(x)\overline{f(x)}\Big]_{-\infty}^{\infty}}_{\text{=0 since f is a Schwartz function}} - \int_{-\infty}^{\infty} f^{'}(x)\,\overline{f'(x)}\,dx \\
&=(f^{'}, f^{'})
\end{aligned} \tag{34}
$$

Recall that Schwartz functions and their derivatives decay to zero at $\pm\infty$. To see why in detail go to the Appendix.

Thus we have that:

$$
\begin{aligned}
(Lf, f) &=(f^{'}, f^{'}) + (x^2 f, f) \\
&= \int_{-\infty}^{\infty} |f^{'}(x)|^2\,dx + \int_{-\infty}^{\infty} x^2\,|f^{'}(x)|^2\,dx
\end{aligned} \tag{35}
$$

Now we can show that:

$$
\int_{-\infty}^{\infty} |f^{'}(x)|^2\,dx = 4\pi^2 \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2\,d\xi \tag{36}
$$

To do this we need the Plancherel theorem: If $g \in \mathcal{S}(\mathbb{R})$ then:

$$
\|\hat{g}\| = \|g\| \tag{37}
$$

where: $\|g\| = \Big(\int_{-\infty}^{\infty} |g(x)|^2\,dx\Big)^{\frac{1}{2}}$. See Theorem 1.12 page 143 [6]. With $g(x) = f^{'}(x)$ and noting that the Fourier transform of $f^{'}(x)$ is $2\pi i\xi\hat{f}(\xi)$ and squaring (37) we have:

$$
\int_{-\infty}^{\infty} |\hat{g}(\xi)|^2\,d\xi = \int_{-\infty}^{\infty} 4\pi^2\xi^2 |\hat{f}(\xi)|^2\,d\xi = 4\pi^2 \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2\,d\xi \tag{38}
$$

Going back to (35) we have:

$$
\begin{aligned}
(Lf, f) &=4\pi^2 \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2\,d\xi + \int_{-\infty}^{\infty} x^2\,|f^{'}(x)|^2\,dx \\
&=\bigg(\Big(2\pi \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2\,d\xi\Big)^{\frac{1}{2}}\bigg)^2 + \bigg(\Big(\int_{-\infty}^{\infty} x^2\,|f^{'}(x)|^2\,dx\Big)^{\frac{1}{2}}\bigg)^2 \\
&\geq 2 \times 2\pi\Big(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2\,d\xi\Big)^{\frac{1}{2}} \Big(\int_{-\infty}^{\infty} x^2\,|f^{'}(x)|^2\,dx\Big)^{\frac{1}{2}} \\
&\geq 4\pi\Big(\frac{1}{16\pi^2}\Big)^{\frac{1}{2}} \\
&=1
\end{aligned} \tag{39}
$$

This shows that $L \geq I$. Note in the derivation in (39) the simple but critical insight is that $a^2 + b^2 \geq 2ab$.

2.6 Exercise 23 part (b)

Given the definition of the operators A, A^* (see (10)-(11)) we have to prove (12)-(14). In what follows, $f, g \in \mathcal{S}(\mathbb{R})$:

(i) We want to show that $(Af, g) = (f, A^*g)$.

$$\begin{aligned}
(Af, g) &= \int_{-\infty}^{\infty} \Big(\frac{df}{dx} + xf(x)\Big)\overline{g(x)}\, dx \\
&= \int_{-\infty}^{\infty} \frac{df}{dx}\, \overline{g(x)}\, dx + \int_{-\infty}^{\infty} xf\, \overline{g(x)}\, dx \\
&= \underbrace{\Big[f(x)\overline{g(x)}\Big]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} f(x)\, \frac{d\bar{g}}{dx}\, dx + \int_{-\infty}^{\infty} xf(x)\overline{g(x)}\, dx \\
&= -\int_{-\infty}^{\infty} f(x)\, \frac{d\bar{g}}{dx}\, dx + \int_{-\infty}^{\infty} xf(x)\overline{g(x)}\, dx
\end{aligned} \tag{40}$$

$$\begin{aligned}
(f, A^*g) &= \int_{-\infty}^{\infty} f(x)\, \overline{\Big(-\frac{dg}{dx} + xg(x)\Big)}\, dx \\
&= -\int_{-\infty}^{\infty} f(x)\, \frac{d\bar{g}}{dx}\, dx + \int_{-\infty}^{\infty} f(x)\, \overline{xg(x)}\, dx \\
&= -\int_{-\infty}^{\infty} f(x)\, \frac{d\bar{g}}{dx}\, dx + \int_{-\infty}^{\infty} xf(x)\overline{g(x)}\, dx \quad \text{since } x = \bar{x} \\
&= (Af, g)
\end{aligned} \tag{41}$$

(ii) We want to show that $(Af, Af) = (A^*Af, f) \geq 0$

$$
\begin{aligned}
(Af, Af) &= \int_{-\infty}^{\infty} \Big(\frac{df}{dx} + xf(x)\Big) \overline{\Big(\frac{df}{dx} + xf(x)\Big)}\, dx \\
&= \int_{-\infty}^{\infty} \Big(\frac{df}{dx} + xf(x)\Big) \Big(\frac{d\bar{f}}{dx} + x\overline{f(x)}\Big)\, dx \qquad \text{noting } x = \bar{x} \\
&= \int_{-\infty}^{\infty} \Big(\frac{df}{dx}\frac{d\bar{f}}{dx} + x\overline{f(x)}\frac{df}{dx} + xf(x)\frac{d\bar{f}}{dx} + x^2 f(x)\overline{f(x)}\Big)\, dx \\
&= \int_{-\infty}^{\infty} \Big[\Big(\frac{df}{dx} + xf(x)\Big)\frac{d\bar{f}}{dx} + x\Big(\frac{df}{dx} + xf(x)\Big)\overline{f(x)}\Big]\, dx \\
&= \underbrace{\Big[\Big(\frac{df}{dx} + xf(x)\Big)\overline{f(x)}\Big]_{-\infty}^{\infty}}_{=0 \text{ by Schwartz property}} - \int_{-\infty}^{\infty} \overline{f(x)}\frac{d}{dx}\Big(\frac{df}{dx} + xf(x)\Big)\, dx + \int_{-\infty}^{\infty} x\Big(\frac{df}{dx} + xf(x)\Big)\overline{f(x)}\, dx \\
&= -\int_{-\infty}^{\infty} \overline{f(x)}\frac{d}{dx}\Big(\frac{df}{dx} + xf(x)\Big)\, dx + \int_{-\infty}^{\infty} x\Big(\frac{df}{dx} + xf(x)\Big)\overline{f(x)}\, dx \\
&= \int_{-\infty}^{\infty} \Big(-\frac{d^2 f}{dx^2} - f(x) + x^2 f(x)\Big)\overline{f(x)}\, dx
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
(A^*Af, f) &= \int_{-\infty}^{\infty} \Big[-\frac{d}{dx}(Af) + xAf\Big]\overline{f(x)}\, dx \\
&= \int_{-\infty}^{\infty} \Big[-\frac{d}{dx}\Big(\frac{df}{dx} + xf(x)\Big) + x\Big(\frac{df}{dx} + xf\Big)\Big]\overline{f(x)}\, dx \\
&= \int_{-\infty}^{\infty} \Big(-\frac{d^2 f}{dx^2} - f(x) + x^2 f(x)\Big)\overline{f(x)}\, dx \\
&= (Af, Af)
\end{aligned}
\tag{43}
$$

Now $(g, g) \geq 0$ as a result of the definition of inner product ie $(g, g) = \int_{-\infty}^{\infty} g(x)\,\overline{g(x)}\, dx = \int_{-\infty}^{\infty} |g(x)|^2\, dx \geq 0$. Let $g = Af$ and we have $(Af, Af) = (A^*Af, f) \geq 0$. We can now show that $L \geq I$:

$$
\begin{aligned}
(Af, Af) &= \int_{-\infty}^{\infty} \Big(-\frac{d^2 f}{dx^2} - f(x) + x^2 f(x)\Big)\overline{f(x)}\, dx \\
&= \int_{-\infty}^{\infty} \Big(-\frac{d^2}{dx^2} - I + x^2\Big) f(x\,\overline{f(x)}\, dx \\
&= \int_{-\infty}^{\infty} (L - I) f(x)\,\overline{f(x)}\, dx \qquad \text{using (7)} \\
&= \geq 0
\end{aligned}
\tag{44}
$$

This implies that $L \geq I$.

(iii) We now want to show that $A^*A = L - I$. That this is the case follows from (43)-(44).

2.7 Exercise 23 part (c)

We have to use the fact that $(A_t^* A_t f, f) \geq 0$ to give another proof of the Heisenberg uncertainty principle which says that whenever $\int_{-\infty}^{\infty} |f(x)|^2 \, dx = 1$ then:

$$\Big(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx \Big) \Big(\int_{-\infty}^{\infty} \left| \frac{df}{dx} \right|^2 dx \Big) \geq \frac{1}{4} \tag{45}$$

$$\begin{aligned}
(A_t^* A_t f, f) &= \int_{-\infty}^{\infty} \Big(-\frac{d}{dx}(A_t f) + tx A_t f \Big) \overline{f(x)} \, dx \\
&= \int_{-\infty}^{\infty} \Big[-\frac{d}{dx}\Big(\frac{df}{dx} + txf(x)\Big) + tx\Big(\frac{df}{dx} + txf(x)\Big) \Big] \overline{f(x)} \, dx \\
&= \int_{-\infty}^{\infty} \Big[-\frac{d^2 f}{dx} - tx\frac{df}{dx} - tf(x) + tx\frac{df}{dx} + t^2 x^2 f(x) \Big] \overline{f(x)} \, dx \\
&= -\,(f^{''}, f) - t\underbrace{(f, f)}_{=1} + t^2 (x^2 f, f) \\
&= t^2 (x^2 f, f) - t - (f^{''}, f) \\
&= t^2 (x^2 f, f) - t - \int_{-\infty}^{\infty} \frac{df^{'}}{dx} \overline{f(x)} \, dx \\
&= t^2 (x^2 f, f) - t - \Bigg[\underbrace{\Big[f^{'}(x) \, \overline{f(x)} \Big]_{-\infty}^{\infty}}_{=0 \text{ Schwartz property}} - \int_{-\infty}^{\infty} f^{'}(x) \, \bar{f}^{'}(x) \Bigg] dx \\
&= t^2 (x^2 f, f) - t + (f^{'}, f^{'}) \\
&\geq 0 \quad \forall t \in \mathbb{R}
\end{aligned} \tag{46}$$

$t^2(x^2 f, f) - t + (f^{'}, f^{'})$ is a quadratic polynomial in t and so the final line of (46) implies:

$$1 - 4(x^2 f, f)\,(f^{'}, f^{'}) \leq 0 \tag{47}$$

Thus we have:

$$\begin{aligned}
(x^2 f, f)\,(f^{'}, f^{'}) &\geq \frac{1}{4} \\
ie \quad \left(\int_{-\infty}^{\infty} x^2 \, |f(x)|^2 \, dx \right) \left(\int_{-\infty}^{\infty} |\frac{df}{dx}|^2 \, dx \right) &\geq \frac{1}{4}
\end{aligned} \tag{48}$$

2.8 Problem 7 part (d) - proof

We have to show that the h_k are eigenfunctions of the operator L (see (7)) and that:

$$Lh_k = (2k+1)h_k \tag{49}$$

and also show mutual orthogonality for the L^2 inner product on Schwartz space,

We proceed as follows:

$$
\begin{aligned}
\sum_{k=0}^{\infty} L(h_k(x))\,\frac{t^k}{k!} =& L\Big(e^{-(\frac{x^2}{2}-2tx+t^2)}\Big)\\
=& -\frac{d^2}{dx^2}e^{-(\frac{x^2}{2}-2tx+t^2)} + x^2\, e^{-(\frac{x^2}{2}-2tx+t^2)}\\
=& -\frac{d}{dx}\Big\{\Big(e^{-(\frac{x^2}{2}-2tx+t^2)}\Big)(-x+2t)\Big\} + x^2\, e^{-(\frac{x^2}{2}-2tx+t^2)}\\
=& e^{-(\frac{x^2}{2}-2tx+t^2)}\Big(-(x+2t)^2-1\Big) + x^2\, e^{-(\frac{x^2}{2}-2tx+t^2)}\\
=& e^{-(\frac{x^2}{2}-2tx+t^2)}(1-4t^2+4tx)\\
=& e^{-(\frac{x^2}{2}-2tx+t^2)} + (4tx-4t^2)e^{-(\frac{x^2}{2}-2tx+t^2)}\\
=& \sum_{k=0}^{\infty} h_k(x)\,\frac{t^k}{k!} + 2t\frac{d}{dt}e^{-(\frac{x^2}{2}-2tx+t^2)}\\
=& \sum_{k=0}^{\infty} h_k(x)\,\frac{t^k}{k!} + 2t\frac{d}{dt}\sum_{k=0}^{\infty} h_k(x)\,\frac{t^k}{k!}\\
=& \sum_{k=0}^{\infty} h_k(x)\,\frac{t^k}{k!} + 2t\sum_{k=0}^{\infty} kh_k(x)\,\frac{t^{k-1}}{k!}\\
=& \sum_{k=0}^{\infty} h_k(x)\,\frac{t^k}{k!} + 2k\sum_{k=0}^{\infty} h_k(x)\,\frac{t^k}{k!}\\
=& \sum_{k=0}^{\infty}(2k+1)h_k(x)\,\frac{t^k}{k!}
\end{aligned}
\tag{50}
$$

Hence $Lh_k = (2k+1)h_k$.

Orthogonality/orthonormality follows from the Hermeticity of the operator. Recall that every Hermitian operator has at least one basis consisting of orthonormal eigenvectors (eg [5],p.26). This problem is not based on a standard quantum mechanical expositon. In that context one would prove that for an operator Ω we have the following with the usual bra and ket notation:

$$
\begin{aligned}
\big(\Omega\big)^{\dagger}_{ij} &= \langle i|\Omega^{\dagger}|j\rangle\\
&= \langle \Omega\, i|j\rangle\\
&= \langle j|\Omega\, i\rangle^{*}\\
&= \langle j|\Omega|i\rangle^{*}
\end{aligned}
\tag{51}
$$

In other words:

$$
\Omega^{\dagger}_{ij} = \Omega^{*}_{ji} \tag{52}
$$

$\Omega^\dagger$ (Omega "dagger") is the adjoint of Ω ie the matrix representing $\Omega^\dagger$ is the transpose conjugate of the matrix representing Ω. An Hemitian operator is one for which:

$$\Omega^\dagger = \Omega \tag{53}$$

(see [5], p. 27)

In engineering contexts it is common to see proofs of orthogonality of Hermite, Laguerre and Legendre polynomials which are based on using generating functions to develop recurrence relations or an integration by parts style of approach. Indeed, in the Appendix I have set out a generating function approach which completely skates across the surface of whether the order of summation and integration can be exchanged. Later I will show why with Schwartz functions it is kosher for integration and summation to be exchanged because we will have uniform convergence of the relevant infinite series. Indeed, Courant and Hilbert use the method of integral transformation to solve the Hermite equation $y^{''} - 2xy^{'} + 2ny = 0$ (see [1], pp 508-9) and they explicity assume the interchange of integration and summation for that method (see [1], p 467].

For the purposes of Problem 7 part (d) the authors want you to use general properties of linear operators in the context of the L^2 inner product on Schwartz space rather than the sort of approach one might find in an engineering textbook. The approach is much closer to the quantum mechanical one given in (51). In the context of the L^2 inner product on Schwartz space the Hermiticity is proved by proving this relation which is the analogue of transposed conjugacy in the matrtix context:

$$(Lf, g) = \overline{(Lg, f)} \tag{54}$$

Remember that the quantum mechanical matrix representation can also be based on linear operators on Hilbert space with an inner product eg for a classical approach which deals with operators in Hilbert space in analogy with the matrix methods see [7] pages 91-96 in particular.

For the LHS of (54) we have:

$$\begin{aligned}
(f, Lg) &= \int_{-\infty}^{\infty} f(x) \overline{\Big(-\frac{d^2 g}{dx^2} + x^2 g(x)\Big)}\, dx \\
&= \int_{-\infty}^{\infty} f(x) \Big(- \frac{d^2 \bar{g}}{dx^2} + x^2 \overline{g(x)}\Big)\, dx \\
&= \underbrace{\Big[- \frac{d\bar{g}}{dx}\frac{df}{dx}\Big]_{-\infty}^{\infty}}_{=0 \text{ Schwartz properties}} + \int_{-\infty}^{\infty} \frac{df}{dx}\frac{d\bar{g}}{dx}\, dx + \int_{-\infty}^{\infty} x^2 f(x)\overline{g(x)}\, dx \\
&= \int_{-\infty}^{\infty} \frac{df}{dx}\frac{d\bar{g}}{dx}\, dx + \int_{-\infty}^{\infty} x^2 f(x)\overline{g(x)}\, dx
\end{aligned} \tag{55}$$

The RHS of (54) is:

$$
\begin{aligned}
\overline{(Lg,f)} =& \overline{\int_{-\infty}^{\infty} \Big(-\frac{d^2g}{dx^2} + x^2 g(x)\Big)\overline{f(x)}\, dx} \\
=& \int_{-\infty}^{\infty} f(x) \Big(-\frac{d^2\bar{g}}{dx^2} + x^2\overline{g(x)}\Big)\, dx \\
=& \int_{-\infty}^{\infty} \frac{df}{dx}\frac{d\bar{g}}{dx}\, dx + \int_{-\infty}^{\infty} x^2 f(x)\overline{g(x)}\, dx
\end{aligned} \tag{56}
$$

So L is Hermitian hence its eigenvectors are orthogonal. Note there is no explicit integration of eigenvectors involved - it is a general property of Hermitian operators

2.9 Problem 7 part (e)

We have to show in this problem that $\int_{-\infty}^{\infty} \big[h_k(x)\big]^2\, dx = \sqrt{\pi}2^k k!$ and the hint is to square the generating function. Thus we have (using (1)):

$$
\begin{aligned}
\Big(\sum_{k=0}^{\infty} h_k(x)\frac{t^k}{k!}\Big)^2 =& e^{-(x^2-4tx+2t^2)} \\
\sum_{k=0}^{\infty} h_k^2(x)\frac{t^{2k}}{(k!)^2} + \sum_{j<k}^{\infty} h_j(x)h_k(x)\frac{t^j}{j!}\frac{t^k}{k!} =& e^{-(x^2-4tx+2t^2)}
\end{aligned} \tag{57}
$$

We now integrate from $-\infty$ to ∞ and assume interchange of integration and summation and the term involving $\int_{-\infty}^{\infty} h_j(x)h_k(x)\, dx$ vanishes because we have established orthgonality. The integration of the term on the RHS of (57) is:

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-(x^2-4tx+2t^2)}\, dx =& \int_{-\infty}^{\infty} e^{-[(x-2t)^2-4t^2+2t^2]}\, dx \\
=& e^{2t^2}\int_{-\infty}^{\infty} e^{-(x-2t)^2}\, dx \\
=& e^{2t^2}\int_{-\infty}^{\infty} e^{-u^2}\, du \quad [u = x-2t] \\
=& \sqrt{\pi}e^{2t^2} \\
=& \sqrt{\pi}\sum_{k=0}^{\infty}\frac{2^k t^{2k}}{k!}
\end{aligned} \tag{58}
$$

Hence we get:

$$
\sum_{k=0}^{\infty}\int_{-\infty}^{\infty} h_k^2(x)\, dx\, \frac{t^{2k}}{(k!)^2} = \sqrt{\pi}\sum_{k=0}^{\infty}\frac{2^k t^{2k}}{k!} \tag{59}
$$

Equating coefficients we get the desired result.

2.10 Proving that Schartz functions allow the interchange of summation and integration

Given the broad global properties of Schwartz functions inherent in their definition (see (20)) it "morally" (to use a verbal flourish of Eli Stein) has to be the case that you can interchange integration and summation - there would be no justice in the universe if you could not! In the context of this problem what we actually need to prove is that integration and summation can be interchanged in this context (see (57)):

$$\int_{-\infty}^{\infty} \sum_{k=0}^{\infty} h_k^2(x) \frac{t^{2k}}{(k!)^2}\, dx = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} h_k^2(x)\, dx\, \frac{t^{2k}}{(k!)^2} \tag{60}$$

Note that we aren't worried about the role of t - it merely performs a formal role in the context of the generating function approach. The other point to note at the outset is that we know that $h_k(x)$ is a Schwartz function for each k and so the product also ought to be Schwartz. That is the case and is direct result of the definition and Leibnitz's rule for the derivative of a product. Thus if f, g are two Schwartz functions then so is $H = fg$. The l^{th} derivative of the product fg is: ($D^j = \frac{d^j}{dx^j}$):

$$D^l(fg) = \sum_{i=0}^{l} \binom{l}{i} D^{l-i} f\, D^i g \tag{61}$$

Thus we have to show that:

$$\sup_{x\in\mathbb{R}} |x|^m \left| H^{(l)}(x) \right| < \infty \quad \forall m, l \geq 0 \tag{62}$$

Now

$$\begin{aligned} |x|^m \left| H^{(l)}(x) \right| &= |x|^m \left| \sum_{i=0}^{l} \binom{l}{i} D^{l-i} f\, D^i g \right| \\ &\leq |x|^m \sum_{i=0}^{l} \binom{l}{i} \left(|D^{l-i} f| + |D^i g| \right) \\ &= \sum_{i=0}^{l} \binom{l}{i} \left(|x|^m\ |D^{l-i} f| + |x|^m\ |D^i g| \right) \end{aligned} \tag{63}$$

Each component on the RHS of the last line in (63) is bounded since f and g are Schwartz functions. Bearing in mind that for bounded functions:

$$\sup\{f(x) + g(x)\} \leq \sup\{f(x)\} + \sup\{g(x)\} \tag{64}$$

we can conclude that $\sup_{x\in\mathbb{R}} |x|^m \left| H^{(l)}(x) \right| < \infty \quad \forall m, l \geq 0$.

The proof of interchange of integration and summation is based on the following theorem and the Weierstrass M-test.

2.10.1 Theorem on interchange of integration and summation

If the series $F(x) = \sum_n f_n(x)$ is uniformly convergent in the interval (a, b) and if each of the functions $f_n(x)$ is continuous on (a, b) then $\int_{c_1}^{c_2} F(x)\,dx = \sum_n \int_{c_1}^{c_2} f_n(x)\,dx$ if $a \leq c_1 < c_2 \leq b$.

2.10.2 Weierstrass M-test

To prove uniform convergence the M-test requires that we prove for all values of $x \in (a, b)$ the functions $f_n(x)$ have the property that $|f_n(x)| \leq M_n$ where M_n is a positive constant independent of x and the series $\sum_n M_n$ converges. The series $\sum_n f_n(x)$ is then uniformly convergent and absolutely convergent.

Now let $H_k(x) = h_k^2(x)$. For every $k = 0, 1, 2, \dots$, H_k is a Schwartz function. This means that for each k there exists a $B_k > 0$ such that:

$$\sup_{x\in\mathbb{R}} |x|^m \left| H_k^{(l)}(x) \right| < B_k \quad \forall m, l \geq 0 \tag{65}$$

We have an infinite set of positive bounds B_k each which is finite since all the functions are Schwartz functions. Thus this infinite set is bounded above and hence has a supremum which we will call $B + 1$ for reasons which will quickly become clear.

We fix k ie fix the particular Hermite functions, and thus with $l = 0$ in (65) we have for any $x \in \mathbb{R}$:

$$|x|^m \, |H_k(x)| < B + 1 \tag{66}$$

But for any x we can choose m so that:

$$|x|^m = (B + 1)^{k+1} \tag{67}$$

Note that in (67) k and B are fixed and the definition (65) allows us to use any m to characterise x in terms of something fixed. Plugging this into (66) we see that:

$$\begin{aligned} (B + 1)^{k+1} \, |H_k(x)| &\leq B + 1 \\ \therefore \; |H_k(x)| &\leq \frac{1}{(B + 1)^k} \end{aligned} \tag{68}$$

Now $\frac{1}{B+1} < 1$ hence $\sum_{k=0}^{\infty} |H_k(x)| \leq \sum_{k=0}^{\infty} \frac{1}{(B+1)^k} < \infty$, Since the series is a convergent geometric series the Weierstrass M-test is satisfied (the $M_k = \frac{1}{(B+1)^k}$ in this case). Note that the M_k do not depend on x and this was achieved by the existence of B and the step in (67).

So we really can interchange integration and summation with Schwartz functions.

3 Appendix

3.1 The exponential grows faster than any power

As noted above, Hardy provides a proof of the proposition that the exponential grows faster than any power ([4], pp 407-8). This is such a fundamental property that it needs to be proved although it is often skated over in calculus and analysis courses. What we are seeking to prove is that e^y tends to infinity with y more rapidly than any power of y ie:

$$\lim_{y\to\infty} \frac{y^\alpha}{e^y} = \lim_{y\to\infty} e^{-y}y^\alpha = 0 \tag{69}$$

for all values of α however great.

We start with this claim (proved in beginning calculus courses) with $\alpha > 0$:

$$\lim_{x\to\infty} x^{-\alpha}\ln x = 0 \tag{70}$$

Let $\beta > 0$ be some rational number. By definition:

$$\ln x = \int_1^x \frac{dt}{t} \tag{71}$$

This inequality holds (because $t > 1$ and $\beta > 0$ and $t^\beta = e^{\beta \ln t}$):

$$t^{-1} < t^\beta t^{-1} = t^{\beta-1} \tag{72}$$

Therefore:

$$\ln x = \int_1^x \frac{dt}{t} < \int_1^x \frac{dt}{t^{1-\beta}} = \frac{x^\beta - 1}{\beta} < \frac{x^\beta}{\beta} \text{ which holds for } x > 1 \tag{73}$$

Since $\alpha > 0$ we can always choose a smaller positive β and then we have:

$$0 < \frac{\ln x}{x^\alpha} < \frac{x^\beta}{\beta x^\alpha} = \frac{x^{\beta-\alpha}}{\beta} \tag{74}$$

But $x^{\beta-\alpha} \to 0$ as $x \to \infty$ since $\beta < \alpha$. Therefore:

$$x^{-\alpha}\ln x \to 0 \text{ as } x \to \infty \tag{75}$$

That this is the case follows from (74) because we have "sandwiched" $\frac{\ln x}{x^\alpha}$ with something which approaches 0 (ie $\frac{x^{\beta-\alpha}}{\beta}$) from the top with 0 on the bottom. This holds for any positive α.

In (75) let $\beta = \frac{1}{\alpha}$ where $\alpha > 0$. Therefore:

$$x^{-\frac{1}{\beta}}(\ln x)^{\alpha\,\beta} = x^{-\frac{\alpha\beta}{\beta}}(\ln x)^{\alpha\beta} = [x^{-1}(\ln x)^\beta]^\alpha \to 0 \tag{76}$$

But this implies that:

$$x^{-1}(\ln x)^{\beta} \to 0. \tag{77}$$

Now let $y = \ln x$ in (77) and we have $e^{-y}y^{\beta} \to 0$ as $y \to \infty$. It is clear from this that $e^{\gamma y}$ tends to ∞ if $\gamma > 0$ and to 0 if $\gamma < 0$ in each case faster than any power of y.

3.2 Why Schwartz functions decay to zero at infinity

The definition of a Schwartz function is given in (20) where we let $k = 1$ and $l = 0$. Thus (20) says for any f:

$$\sup_{x \in \mathbb{R}} |x|\big|f(x)\big| < \infty \tag{78}$$

This means that $|x|\big|f(x)\big|$ is bounded ie there exists a $B > 0$ such that:

$$|x|\big|f(x)\big| < B \quad \forall x \in \mathbb{R} \tag{79}$$

If you give me an $\epsilon > 0$ I can find an $M > 0$ such that $|f(x)| < \frac{B}{M} < \epsilon$ for all $|x| > M$. This means that $|f(x)| \to 0$ as $|x| \to \infty$.

The same logic applies when derivatives are involved eg $|x|\big|f^{'}(x)\big|$

3.3 Orthogonality of Hermite functions ignoring uniform convergence issues

In this hand waving proof we simply assume that integration and summation can be exchanged (and they certainly can be for Schwartz functions so it is actually all kosher in the end) and do the relevant integrals.

We start with:

$$\begin{aligned} \sum_{k=0}^{\infty} h_k(x)\frac{s^k}{k!} &= e^{-(\frac{x^2}{2}-2sx+s^2)} \\ \sum_{l=0}^{\infty} h_l(x)\frac{t^l}{l!} &= e^{-(\frac{x^2}{2}-2tx+t^2)} \end{aligned} \tag{80}$$

Multiplying the functions in (80) we get:

$$e^{-(x^2-2(s+t)x+s^2+t^2)} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h_k(x)\, h_l(x)\, \frac{s^k}{k!}\frac{t^l}{l!} \tag{81}$$

Now throw caution to the winds and integrate:

$$\int_{-\infty}^{\infty} e^{-\left[(x-(s+t))^2-2st\right]}\, dx = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{s^k}{k!}\frac{t^l}{l!} \int_{-\infty}^{\infty} h_k(x)\, h_l(x)\, dx \tag{82}$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-\left[(x-(s+t))^2-2st\right]}\,dx =& e^{2st}\int_{-\infty}^{\infty} e^{-(x-(s+t))^2}\,dx \\
=& e^{2st}\int_{-\infty}^{\infty} e^{-u^2}\,du \quad [u = x-(s+t)] \\
=& \sqrt{\pi}e^{2st} \\
=& \sqrt{\pi}\sum_{k=0}^{\infty}\frac{2^k s^k t^k}{k!}
\end{aligned} \tag{83}
$$

Now we just equate coefficients in (81) and the last line of (83):

$$
\int_{-\infty}^{\infty} h_k(x)\,h_l(x)\,dx = \begin{cases} \sqrt{\pi}2^k k! & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases} \tag{84}
$$

4 References

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5 History

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