Induction Problem Set Solutions

These problems flow on from the larger theoretical work titled "Mathematical induction - a miscellany of theory, history and technique - Theory and applications for advanced secondary students and first year undergraduates"

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1. Two easy ones to warm up

(1) Prove by induction \( S_n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \) for integers \( n \geq 1 \)

Gauss implicitly derived this formula in his head when in primary school. His teacher (apparently a bit of a brute) asked the children to sum the first 100 numbers. Gauss noticed this pattern:

\[
\begin{array}{cccccc}
1 & 2 & 3 & \ldots & & 100 \\
100 & 99 & 98 & \ldots & & 1 \\
101 & 101 & 101 & \ldots & & 101 \\
\end{array}
\]

\[
\begin{array}{cccccc}
100 & 99 & 98 & \ldots & & 1 \\
101 & 101 & 101 & \ldots & & 101 \\
\end{array}
\]

Since there are 100 pairs equal to 101 the sum must be 100 x 101 / 2 = 5050 (you have to divide by 2 to negate the duplication).

There is an amusing fictional book which casts Gauss in a somewhat different light. The book is: Daniel Kehlmann, "Measuring the World", First Vintage Books, 2007 and deals with the exploits of Baron von Humboldt and Gauss. After reading this book Gauss’ aura is, well, less pristine than it would otherwise have been. The thought of him elbowing his way to the food at a function and technique - Theory and applications for advanced secondary students and first year undergraduates"
Then assume the formula is true for any n.

(1) Prove that \( F_n = F_{n+2} - 1 \)

**Solution:**
The formula is true for \( n = 1 \) since \( F_1 = 1 \), \( F_2 = 1 \) and \( F_3 = 1 \) - 1 = 1 + 1 - 1 = 1

Suppose the formula is true for any \( n \).

Then \( F_1 + F_2 + F_3 + \ldots + F_n + F_{n+1} = F_{n+2} - 1 + F_{n+1} \) using the induction hypothesis.

\[ = F_{n+3} - 1 \text{ (since } F_{n+3} = F_{n+2} + F_{n+1} \text{)} \]

Accordingly the formula is true for all \( n \).

(2) Prove that \( F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n} \)

**Solution:**
The formula is true for \( n = 1 \) since LHS = \( F_1 = 1 \) and RHS = \( F_2 = 1 \)

Suppose the formula is true for any \( n \).

Then \( F_1 + F_3 + F_5 + \ldots + F_{2n-1} + F_{2n+1} = F_{2n} + F_{2n+1} \) using the induction hypothesis

\[ = F_{2n-1} + 1 \text{ using the definition of } F_n \]

So the formula is true for \( n + 1 \) and hence is true for all \( n \).

(3) Prove that \( F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1 \)

**Solution:**
The proof follows the approach of (2).

The formula is true for \( n = 1 \) since LHS = \( F_2 = 1 \) and RHS = \( F_3 - 1 = 2 - 1 = 1 \)

Assume the formula is true for any \( n \) then:

\[ F_2 + F_4 + F_6 + \ldots + F_{2n} + F_{2n+2} = F_{2n+1} - 1 + F_{2n+2} \text{ using the induction hypothesis} \]

\[ = F_{2n+3} - 1 \text{ using the definition of } F_n \]

So the formula is true for \( n + 1 \) and hence true for all \( n \).

(4) Prove that \( F_{n+1}^2 - F_n F_{n+2} = (-1)^n \)

**Solution:**
When \( n = 1 \) LHS = \( F_2^2 - F_1 F_3 = 1^2 - 1 \cdot 2 = (-1)^1 \) so the formula is true for \( n = 1 \).

Assume the formula is true for any \( n \).

Then \( F_{n+1}^2 - F_n F_{n+3} = F_{n+2}^2 - F_{n+1} (F_{n+2} + F_{n+1}) \) using the definition of \( F_n \)

\[ = F_{n+2}^2 - F_{n+3} F_{n+1}^2 \]
relatively prime. There is a stronger result that can be proved and it is this:

\[ F_{n+1}^2 - F_{n+1} + 1 = \alpha^{n+1} - \beta^{n+1} \]

Thus \( d > 1 \). Thus for this \( n \), \( d \mid F_n \) for some positive integer \( r \). Hence, \( d \mid F_{n+1} \). Since \( d \mid F_n \) we have established that \( d > 1 \) is also a divisor of \( F_{n+1} \) and we can apply the same logic to descend (see the discussion in the main paper about the method of descent) further down the chain until we hit \( F_1 = 1 \) at the bottom. Since \( d \) cannot divide 1 we have our contradiction and hence all consecutive Fibonacci numbers are relatively prime. There is a stronger result that can be proved and it is this:

\[ \text{gcd}(F_n, F_{n+1}) = 1 \] Hint: Think proof by contradiction.

(7) Conjecture a formula for \( F_n^2 + F_{n-1}^2 \) Hint: think of the formula using the Golden Ratio (See the main work)

Solution:

Recall that \( \alpha = \frac{1 + \sqrt{5}}{2} \) and \( \beta = \frac{1 - \sqrt{5}}{2} \) so that \( \alpha \beta = -1 \), \( \alpha - \beta = \sqrt{5} \) and \( \alpha + \beta = 1 \)

Using this formula \( F_n^2 + F_{n-1}^2 = \frac{\alpha^{2n+2} - 2(\alpha n) + \beta^{2n+2} - 2\beta n - 2(\beta n) - 2(\alpha n)}{5} \)

\[ = \frac{\alpha^{2n+2} - 2(\alpha n) + \beta^{2n+2} - 2\beta n - 2(\alpha n)}{5} \]

\[ = \frac{\alpha^{2n+2} + \beta^{2n+2} - 2(\alpha n) - 2\beta n - 2(\alpha n)}{5} \]

\[ = \frac{\alpha^{2n+2} + \beta^{2n+2} - 2(\alpha n) - 2\beta n - 2(\alpha n)}{5} \]

\[ = \frac{(\alpha^{n+1} - \beta^{n+1})(\alpha - \beta) + \alpha^{n+1} + \beta^{n+1}}{5} \]

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\[
\frac{(a^{n+1} - \beta^{n+1}) \sqrt{5} + (1 + a\beta) \beta^{n+1} - (1 + a\beta) a^{n+1}}{5} \quad \text{since} \quad 1 + a\beta = 0
\]

\[
= \left(\frac{a^{n+1} - \beta^{n+1}}{5}\right) \sqrt{5}
\]

\[
= \frac{a^{2n+1} - \beta^{2n+1}}{\sqrt{5}}
\]

\[
= F_{2n-1}
\]

Hence the conjecture is: \( F_n^2 + F_{n-1}^2 = F_{2n-1} \)

(8) Conjecture a formula for \( F_{n+1} F_n + F_n F_{n-1} \) Hint: think of the formula using the Golden Ratio

**Solution:**

We proceed as in (7).

\[
F_{n+1} F_n + F_n F_{n-1} = \frac{(a^{n+1} - \beta^{n+1})(a^n - \beta^n) + (a^n - \beta^n)(a^{n+1} - \beta^{n+1})}{5}
\]

\[
= \frac{a^{2n+1} - \beta^{2n+1} + a^{2n} - \beta^{2n} + a^{2n+1} - \beta^{2n+1}}{5}
\]

\[
= \frac{a^{2n+1} - a (a^n - \beta^n) + a^{2n} - (a^n - \beta^n) + a^{2n+1} - (a^n - \beta^n) + \beta^{2n+1}}{5}
\]

\[
= \frac{a^{2n+1} + \beta^{2n+1} + a^{2n} + \beta^{2n}}{5}
\]

\[
= \frac{a^n - \beta^n}{5} \text{ since } a^n - \beta^n = 0
\]

\[
= \frac{a^2 - \beta^2}{\sqrt{5}}
\]

\[
= F_{2n}
\]

Hence our conjecture is \( F_{n+1} F_n + F_n F_{n-1} = F_{2n} \)

(9) Using (7) and (8) develop an inductive argument that will establish the validity of both of your conjectures.

**Solution:**

Let's start with \( F_{n+1} F_n + F_n F_{n-1} = F_{2n} \), which is true for \( n = 1 \) since \( F_2 F_1 + F_1 F_0 = 1.1 + 1.0 = 1 \) and \( F_{2} = 1 \). If we consider \( F_{n+2} F_{n+1} + F_{n+1} F_n \), we can use the approach in (8) where we replace \( n \) with \( N = n + 1 \) and it is clear that we will end up with \( F_{2n} = F_{2n-2} \) so that the conjecture is true for \( n+1 \). If you grind remorselessly through the algebra of \( F_{n+2} F_{n+1} + F_{n+1} F_n \), you will indeed get equality with \( F_{2n} \).

In essence we know that the Golden Ratio formula is true for all \( n \) (proved in the main work) so all we are doing is some algebraic manipulations with something we know to be true.

A similar approach can be followed with (7) after establishing the base case with \( n = 1 \) i.e. \( F_1^2 + F_0^2 = 1 + 0 = 1 \) and \( F_{2-1} = F_1 = 1 \)

There is a different solution given to (9) in L. Lovász, J. Pelikán and K. Vesztergombi, "Discrete Mathematics: Elementary and Beyond", Springer, 2003, pages 68-69. The authors pull the formulas for (7) and (8) out of the air and then use "simultaneous induction" to get the result. Since Lovasz in particular is a very well known author on substantial combinatorial problems it is worth having a look at the alternative solution. Their proof runs like this:

First they establish \( F_{n+1} F_n + F_n F_{n-1} = F_{2n} \) by using the basic formula \( F_{n+1} = F_n + F_{n-1} \) this way:

\[
F_{n+1} F_n + F_n F_{n-1} = (F_n + F_{n-1}) F_n + (F_{n-1} + F_{n-2}) F_{n-1}
\]

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= (F_n^2 + F_{n-2}^2) + F_n F_{n-1} + F_{n-1} F_{n-2} \quad \text{then using (7) and (8)}

= F_{2n-1} + F_{2n-2}

= F_{2n}, \text{ using the basic formula}

They then prove (7) this way:

\[ F_n^2 + F_{n-1}^2 = (F_n + F_{n-1})^2 = F_n^2 + F_{n-1}^2 + 2F_n F_{n-1} + F_{n-1}^2 \]

\[ = (F_{n-1}^2 + F_{n-2}^2) + F_{n-1}(F_{n-2} + F_{n-1}) + F_{n-1} F_{n-2} \]

\[ = (F_{n-2}^2 + F_{n-3}^2) + F_{n-1} F_{n-2} + F_{n-1} F_{n-2} \]

\[ = F_{2n-3} + F_{2n-2} \]

\[ = F_{2n-1} \]

The authors note that they have used (8) (4.3 in their book) in the proof of (7) (4.2 in their book) and (7) in the proof of (8) and this is fine because two induction proofs have to go simultaneously. If we know that both (7) and (8) are true for a certain value of \( n \), then we prove (7) for the next value (if you look at the proof, you can see that it uses smaller values of \( n \) only), and then use this and the induction hypothesis again to prove (8)

My problem with that type of proof is that it is easy to get lost and while it is not logically flawed it would require discipline to execute accurately.

(10) Continued Fractions

The theory of continued fractions underpins many of the results on Fibonacci sequences. Indeed, in G H Hardy & E M Wright, "An Introduction to the Theory of Numbers," Fifth Edition, Oxford University Press, 1979 the authors develop Fibonacci theory as a subset of a more general treatment of continued fractions. However, to do so requires a reasonable amount of overhead. To get more of an "entry level" feel for how the theory of continued fractions is relevant we need to establish some building blocks and the following series of problems sets out the relevant building blocks in a systematic fashion.

First, if you have not encountered the concept of a continued fraction it is represented as a sequence of numbers of the following form:

\[ 1, 1 + \frac{1}{2}, 1 + \frac{1}{2 + \frac{1}{2}}, 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}, \ldots \]

and so on. There is a standard notation for these continued fractions: \([1],[1,2],[1,2,2],[1,2,2,2],[1,2,2,2,2],[1,2,2,2,2,2],[1,2,2,2,2,2,2]\) etc.

(i) As a first step find \([1],[1,2],[1,2,2],[1,2,2,2],[1,2,2,2,2],[1,2,2,2,2,2],[1,2,2,2,2,2,2]\) in fractional form.

**Solution:**

\[ [1] = 1 \]

\[ [1,2] = 1 + \frac{1}{2} = \frac{3}{2} \]

\[ [1,2,2] = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{2}{5} = \frac{7}{5} \]

\[ [1,2,2,2] = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = 1 + \frac{5}{12} = \frac{17}{12} \]

\[ [1,2,2,2,2] = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} = 1 + \frac{12}{29} = \frac{41}{29} \]

(ii) Show how \( \sqrt{2} \) can be written as a continued fraction: Hint: Start with \( \sqrt{2} = 1 + (\sqrt{2} - 1) \) and it should emerge.

**Solution:**

Using the hint we have that \( \sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + (\sqrt{2} - 1) \left( \frac{\sqrt{2} + 1}{\sqrt{2} + 1} \right) = 1 + \frac{1}{\sqrt{2} + 1} = 1 + \frac{1}{2 + (\sqrt{2} - 1)} = 1 + \frac{1}{2 + \left( \frac{1}{\sqrt{2} - 1} \right)} = 1 + \frac{1}{2 + \left( \frac{1}{\sqrt{2} - 1} \right)} = \left( \frac{1}{2 + \sqrt{2} - 1} \right) \) etc.
This suggests that \( \sqrt{2} = [1,2,2,2,...] = [1,2] \) where the overhead dot signifies infinite repetition.

(iii) Generalise the concept as follows. This is a classic case of where notation is important. First note that \([a_0] = \frac{a_0}{1}\) and then \([a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}\) and \([a_0, a_1, a_2] = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_0 a_1 a_2 + a_0 a_2 + a_1}{a_1 a_2 + 1}\). Show by an inductive argument that:

(a) \([a_0, a_1, ..., a_{n-1}, a_n] = [a_0, [a_1, ..., a_{n-1}, a_n]]\)

**Solution:**

The claim is trivially true for \(n=0\). For \(n=1\) we have that \([a_0, a_1] = [a_0, [a_1]]\). It is given that \([a_0, a_1] = \frac{a_0 a_1 + 1}{a_1}\) and \([a_0, [a_1]]\) holds for all \(n\) by (a) above.

Assume that for any positive integral \(k\) \(n:\)

\([a_0, a_1, ..., a_{k-1}, a_k] = [a_0, [a_1, ..., a_{k-1}, a_k]]\)

Now \([a_0, a_1, ..., a_{n-1}, a_n, a_{n+1}] = [a_0, a_1, [a_2, ..., a_{n-1}, a_n, a_{n+1}]]\) using the induction hypothesis on \([a_1, ..., a_{n-1}, a_n, a_{n+1}]\) because it has \(n+1\) terms. Note that after \(n\) applications of the induction hypothesis to \([a_1, ..., a_{n-1}, a_n, a_{n+1}]\) and its descendants you will strip away the internal brackets. The induction hypothesis justifies this because we assumed that one can reduce the scope of the internal brackets by one term for each \(k\) \(n\).

Thus we get that \([a_0, [a_1, ..., a_{n-1}, a_n, a_{n+1}]] = [a_0, a_1, ..., a_{n-1}, a_n, a_{n+1}]\) so the assertion is true for \(n+1\) and because it is true for \(n=0\) and \(n=1\) it is true for all \(n\).

(b) \([a_0, a_1, ..., a_{n-2}, a_n] = [a_0, a_1, ..., a_{n-2}, a_n + \frac{1}{a_1}]\)

**Solution:**

When \(n=1\), \(LHS = [a_0, a_1] = a_0 + \frac{1}{a_1}\) and \(RHS = [a_0 + \frac{1}{a_1}] = a_0 + \frac{1}{a_1}\) so the proposition is true for \(n=1\). Assume that \([a_0, a_1, ..., a_{n-1}, a_n] = [a_0, a_1, ..., a_{n-2}, a_n + \frac{1}{a_1}]\) holds for all positive integers \(n\).

Then \([a_0, a_1, ..., a_{n-1}, a_n, a_{n+1}] = [a_0, [a_1, ..., a_{n-1}, a_n, a_{n+1}]]\) using (a) above

\([a_0, [a_1, ..., a_{n-1}, a_n + \frac{1}{a_1}]\]

using the induction hypothesis (note that \([a_1, ..., a_{n-1}, a_n + \frac{1}{a_1}]\) has \(n\) terms)

\([a_0, a_1, ..., a_{n-1}, a_n + \frac{1}{a_1}]\)

using (a) again.

Hence the assertion is true for \(n+1\) and is true by induction.

(c) \([a_0, a_1, ..., a_{n-1}, a_n]\) is defined to be the \(n\)th convergent to \([a_0, a_1, ..., a_{n-1}, a_n]\) for \(0 \leq n \leq N\). The name of the game is to get a rational expression for \([a_0, a_1, ..., a_{n-1}, a_n]\). To this end define \(p_n\) and \(q_n\) as follows:

\(p_0 = a_0\), \(p_1 = a_0 a_1 + 1\), \(p_n = a_0 p_{n-1} + p_{n-2}\) for \(2 \leq n \leq N\) and symmetrically,

\(q_0 = 1\), \(q_1 = a_1\), \(q_n = a_n q_{n-1} + q_{n-2}\) for \(2 \leq n \leq N\)

Then \([a_0, a_1, ..., a_{n-1}, a_n] = \frac{p_n}{q_n}\). Prove this assertion by induction. You will need to rely upon (b) above.

**Solution:**

When \(n=0\) we have \([a_0] = a_0\) and \(\frac{p_0}{q_0} = \frac{a_0}{1}\) from the definition. Also when \(n=1\), \([a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}\) while \(\frac{p_1}{q_1} = \frac{a_0 a_1 + 1}{a_1}\). Hence the proposition is true for \(n=0\) and \(n=1\). As our inductive hypothesis we assume that it is true for \(n \geq m\) where \(m < N\), \([a_0, a_1, ..., a_{m-1}, a_m] = \frac{p_m}{q_m}\) \(\frac{a_0 a_{m+1} + p_{m+2}}{a_1 a_m + q_{m+2}}\) where \(p_m, p_{m+2}, q_{m+2}\) depend only on \(a_0, a_1, ..., a_{m-1}\).

Now \([a_0, a_1, ..., a_{m-1}, a_m, a_{m+1}] = [a_0, a_1, ..., a_{m-1}, a_m + \frac{1}{a_{m+1}}]\) using (b) (note here that \([a_0, a_1, ..., a_{m-1}, a_m + \frac{1}{a_{m+1}}]\) has \(m+1\) terms as does \([a_0, a_1, ..., a_{m-1}, a_m]\)

\(\frac{a_0 a_{m+2} + p_{m+2}}{a_1 a_m + q_{m+2}}\) using the induction hypothesis

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Somehow we have to get the RHS to equal 1.

Assume it is true for any n.

Solution:

(i) Prove by induction for integers n \( \geq 1 \):

\[
1 + 2 + 3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2
\]

(ii) When you have got that one under your belt do the same for 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}[n(n+1)]^2 for integral n \( \geq 1 \)

These formulas have been pulled from out of nowhere so it is worth exploring a systematic way of deriving them and the next group of problems deals with the issues relating to that process. The "method of differences" which goes back to Newton (and perhaps beyond) can be used to establish some basic results. The basic idea is that if we are given the sequence \( \{a_n\} \) and we can write each \( a_k \) as a difference ie \( a_k = v_{k+1} - v_k \) then our sum \( \sum_{n=1}^k a_k = \sum_{n=1}^k (v_{k+1} - v_k) = v_{n+1} - v_1 \)

(iii) With this tidbit of knowledge find a formula for 1.2...k + 2.3...(k+1) + \ldots + n(n+1)...(n+k - 1) where k is any positive integer and prove it by induction to confirm the accuracy of your work.

(iv) Show that \( \sum_{r=1}^n r(r + 1) = \frac{n(n+1)(n+2)}{3} \)

(v) Use (iv) to obtain the formula for \( \sum_{n=1}^k r^2 \)

(vi) Use (iii) to establish the formula in (ii).

Solution:

(i) Clearly the proposition is true for n = 1. It is more informatively true for n = 2 ie \( 1^3 + 2^3 = 3^2 \)

Assume it is true for any n.

\( 1^3 + 2^3 + 3^3 + \ldots + n^3 + (n + 1)^3 = (1 + 2 + 3 + \ldots + n)^2 + (n + 1)^3 \) using the induction hypothesis

Somehow we have to get the RHS to equal \( (1 + 2 + 3 + \ldots + n + n + 1)^2 \)

Hence the proposition is established by induction.

(d) In the theory of the approximation of irrational numbers by rational numbers, the following formula arises:

\[
p_n q_{n-1} - p_{n-1} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1} (a_n q_{n-1} + q_{n-2}) = (p_{n-1} q_{n-2} - p_{n-2} q_{n-1})
\]

This suggests a standard inductive approach, namely, replace n with n-1, n-2, ..., 2 so that in the final step one will have \( p_1 q_0 - p_0 q_1 \) with the appropriate sign. Thus:

\[
p_n q_{n-1} - p_{n-1} q_n = - (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) = (-1)^4 (p_{n-2} q_{n-3} - p_{n-3} q_{n-2}) = \ldots = (-1)^n (p_1 q_0 - p_0 q_1) = (-1)^n \]

Now this formula is true for n = 1 since \( p_1 q_0 - p_0 q_1 = a_0 a_1 + 1 - a_0 a_1 = 1 = (-1)^1 \)

Assume it is true for any n.

Then \( p_{n+1} q_n - p_n q_{n+1} = (a_{n+1} p_n + p_{n-1}) q_n - p_n (a_{n+1} q_n + q_{n-1}) \)

\( = - (p_n q_{n-1} - p_{n-1} q_n) \)

\( = (-1)^n \)

Hence the formula is true for n+1 and is established by the principle of induction.

Finally \( \frac{p_{n+1}}{q_n} \frac{p_n}{q_{n+1}} = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n q_{n+1}} = (-1)^n \frac{q_n}{q_{n+1}} \)

3. A subtle extension

(i) Prove by induction for integers n \( \geq 1 \): 1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2

(ii) These formulas have been pulled from out of nowhere so it is worth exploring a systematic way of deriving them and the next group of problems deals with the issues relating to that process. The "method of differences" which goes back to Newton (and perhaps beyond) can be used to establish some basic results. The basic idea is that if we are given the sequence \( \{a_n\} \) and we can write each \( a_k \) as a difference ie \( a_k = v_{k+1} - v_k \) then our sum \( \sum_{k=1}^n a_k = \sum_{k=1}^n (v_{k+1} - v_k) = v_{n+1} - v_1 \)

(iii) With this tidbit of knowledge find a formula for 1.2...k + 2.3...(k+1) + \ldots + n(n+1)...(n+k - 1) where k is any positive integer and prove it by induction to confirm the accuracy of your work.

(iv) Show that \( \sum_{r=1}^n r(r + 1) = \frac{n(n+1)(n+2)}{3} \)

(v) Use (iv) to obtain the formula for \( \sum_{n=1}^k r^2 \)

(vi) Use (iii) to establish the formula in (ii).

Solution:

(i) Clearly the proposition is true for n = 1. It is more informatively true for n = 2 ie 1^3 + 2^3 = 3^2

Assume it is true for any n.

\( 1^3 + 2^3 + 3^3 + \ldots + n^3 + (n + 1)^3 = (1 + 2 + 3 + \ldots + n)^2 + (n + 1)^3 \) using the induction hypothesis

Somehow we have to get the RHS to equal \( (1 + 2 + 3 + \ldots + n + n + 1)^2 \)

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Let’s look at how the answer might play out.

\[(1 + 2 + 3 + ... + n + n + 1)^2 = \left( \sum_{i=1}^{n} i + n + 1 \right)^2 \]

\[= \left( \sum_{i=1}^{n} i \right)^2 + 2(n + 1) \sum_{i=1}^{n} i \cdot (n + 1)^2 \]

\[= \left( \sum_{i=1}^{n} i \right)^2 + 2(n + 1) \frac{n(n+1)}{2} + (n + 1)^2 \]

\[= \left( \sum_{i=1}^{n} i \right)^2 + (n + 1)^2 n \cdot (n + 1)^2 \]

\[= \left( \sum_{i=1}^{n} i \right)^2 + (n + 1)^2 + (n + 1)^3 \]

Now this looks suspiciously like what we want ie:

\[1^3 + 2^3 + 3^3 + ... + n^3 + (n + 1)^3 = (1 + 2 + 3 + ... + n)^2 + (n + 1)^3 = (1 + 2 + ... + n + n + 1)^2 \]

The equality shows that the formula holds for n+1 and hence it is true for all n.

(ii) The next proof relies upon (i)

\[1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n(n+1)^2 \] is true for n=1 since \[1 = \frac{1}{4}(1.2)^2 \]

Assume the proposition is true for any n.

Then \[1^3 + 2^3 + 3^3 + ... + n^3 + (n + 1)^3 = (1 + 2 + 3 + ... + n)^2 + (n + 1)^3 \]

\[= \left( \frac{n(n+1)}{2} \right)^2 + (n + 1)^3 \]

\[= \frac{(n+1)^2(n^2 + 4n + 4)}{4} \]

\[= \frac{(n+1)^2(n+2)^2}{4} \]

ie the formula holds for n+1 and hence is proved for all n by induction

(iii) Find a formula for 1.2..k + 2.3...(k+1) + ... + n(n+1)...(n+k-1) where k is any positive integer and prove it by induction to confirm the accuracy of your guess.

The rth term is \(a_r = r(r+1)...(r+k-1)\) and we need to write this as a difference \(v_{r+1} - v_r\). The following difference will work:

\[v_{r+1} - v_r = \sum_{r=1}^{n} \frac{r(r+1)...(r+k-1)}{k+1} \]

Hence \[1.2..k + 2.3...(k+1) + ... + n(n+1)...(n+k-1) = (v_2 - v_1) + (v_3 - v_2) + ... + (v_{n+1} - v_n) \]

Note that \(\frac{n(n+1)...(n+k-1)}{k+1}\) involves taking the last term of the series and putting the extra factor \((n+k)\) in and then divided by \(k+1\) which is the number of factors (can you see why?).

Hence \[\sum_{r=1}^{n} r(r+1)...(r+k-1) = \frac{n(n+1)...(n+k-1)}{k+1} \]

To prove this by induction take k as given and consider the base case of n = 1. Then \[\sum_{r=1}^{n} r(r+1)...(r+k-1) = 1.2...k \] and \[\frac{n(n+1)...(n+k-1)}{k+1} = \frac{12...(1+k-1)(1+k)}{k+1} = 1.2...k \]. Thus the base case is established. Now assume that the proposition is true for any n.

Then \[\sum_{r=1}^{n} r(r+1)...(r+k-1) = \frac{n(n+1)...(n+k-1)}{k+1} + (n+1)(n+2)...(n+k-1)(n+k) \]

using the induction hypothesis.


\[ \sum_{k=1}^{n+1} r^k \text{ is the right structure for } n+1. \]

Note that there are still \( k+1 \) factors since \( n+k+1 \cdot (n+1)+1 = k+1. \)

(iv) In (iii) take \( k=2 \) the the \( r^k \) term is \( \frac{(n+1)(n+2)-r(r+1)}{3} = r(r+1). \) Thus we can immediately deduce that \( \sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3} \)

\[ \sum_{r=1}^{n} r^3 = \sum_{r=1}^{n} r(r+1) \cdot \sum_{r=1}^{n} r = \frac{n(n+1)(n+2)}{3} \cdot \frac{n(n+1)(n+2)}{2} = \frac{n(n+1)(2n+1)}{6} \]

(vi) We want \( \sum_{r=1}^{n} r^2 \) and we can break \( r^2 \) into the following components: \( r^2 = r(r+1) \cdot 3r - 3r + 1 \)

\[ \text{Hence } \sum_{r=1}^{n} r^2 = \sum_{r=1}^{n} r(r+1) \cdot 3r(r+1) + \sum_{r=1}^{n} r = \]

\[ n(n+1)(n+2)(n+3) - 3n(n+1)(n+2)+ \frac{n(n+1)(2n+1)}{2} = \frac{n(n+1)(2n+1)}{4} \]

In principle it is possible to continue in this vein for sums of the form \( \sum_{r=1}^{n} r^k \) but other methods are employed. For instance it can be shown that \( \sum_{r=1}^{n} r^k = a_{r+1}(r) \cdot \frac{n^{r+1}}{r+1} + a_r(r) \cdot \frac{n^{r+1}}{r} + \ldots + a_1(r) \cdot n \) where the coefficients depend on \( r \) and the constant term is 0 for all such polynomials and the sum of coefficients of each polynomial is 1 (see for instance, Russell A. Gordon, "Some Integrals Involving the Cantor Function", "The American Mathematical Monthly", Volume 116, Number 3, March 2009, page 219.)

### 4. Application of summation to Riemann integration

(i) As a first step to this problem prove the following by induction: \( 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \) for integral \( n \)

(ii) Having got that under your belt let’s see how to apply it to the following integral: \( \int_1^\infty \frac{1}{x} \, dx \) using the theory of Riemann integration. A quick history of Riemann integration is as follows. We suppose that \( f: [a,b] \to \mathbb{R} \) is a bounded function (ie for all \( x \in [a,b], |f(x)| \leq M \) for some \( M > 0 \). Then there is the concept of a partition \( P \) of how you divide up the domain of \( f \) to form little rectangles. A partition is a set \( P = \{a_0, a_1, \ldots, a_n\} \) where \( a_0 < a_1 < \ldots, a_n = b \)

With this partition (which is completely general in principle ie it doesn't have to be nice equal partition) you then consider the upper and lower Riemann sums respectively:

\[ U(P,f) = \sum_{i=1}^{n} M_i \Delta a_i \]

\[ L(P,f) = \sum_{i=1}^{n} m_i \Delta a_i \]

where \( M_i = \sup \{ f(x) : a_{i-1} \leq x \leq a_i \}, m_i = \inf \{ f(x) : a_{i-1} \leq x \leq a_i \} \) and \( \Delta a_i = a_i - a_{i-1} \).

If you don't know what sup and inf are go to the appendix for a brief explanation.

The problem is this: Prove that \( U(P,f) = L(P,f) = \frac{2}{3} \)

[Hint: You might want to consider a partition like \( \left\{ \frac{1}{n} \right\} \) so you can use (i)]

**Solution:**

(i) The proposition is clearly true for \( n=1 \) since \( 1^2 = \frac{1 \times 2 \times 3}{6} \)

Assume the proposition is true for any \( n \) and consider:

\[ 1^2 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \]

using the induction hypothesis

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This demonstrates that the formula is true for \( n+1 \)

(ii) You have to find the upper and lower limits of the Riemann sum relating to \( \int_{1}^{4} \sqrt{x} \, dx \). The hint suggests that the following partition might be useful (go back and look at the definition and note that \( f(x) = \sqrt{x} \))

\[ P = \{ 0, (\frac{1}{n})^2, (\frac{2}{n})^2, \ldots, (\frac{i}{n})^2, \ldots, 1 \} \]

With this partition \( U(P_n,f) = \sum_{i=1}^{n} \left( \frac{i-1}{n} \right)^2 \left( \frac{i}{n} \right)^2 - \left( \frac{i-1}{n} \right)^2 \) = \( \frac{1}{n^2} \sum_{i=1}^{n} (2i^2 - i) \)

Note here that \( M_i = \sup \left\{ \sqrt{x} : \left( \frac{i-1}{n} \right)^2 < x < \left( \frac{i}{n} \right)^2 \right\} = \frac{i-1}{n} \)

and \( m_i = \inf \left\{ \sqrt{x} : \left( \frac{i-1}{n} \right)^2 < x < \left( \frac{i}{n} \right)^2 \right\} = \frac{i-1}{n} \)

Now using (i)

\[ \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - i) = \frac{1}{n^3} \left\{ 2 \frac{n(n+1)}{6} \left( \frac{2n+1}{6} - 3 \frac{n+1}{6} \right) \right\} = \frac{(n+1)(n+2)(n+3)}{6n^3} \]

\[ = \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2} \rightarrow \frac{2}{3} \text{ as } n \rightarrow \infty \]

Working out \( L(P_n,f) \) follows the same logic:

\[ L(P_n,f) = \sum_{i=1}^{n} \left( \frac{i-1}{n} \right)^2 \left( \frac{i}{n} \right)^2 - \left( \frac{i-1}{n} \right)^2 \] = \( \frac{1}{n^2} \sum_{i=1}^{n} (2i^2 - 3i + 1) = \frac{1}{6n^2} \sum_{i=1}^{n} (2n(n+1)(2n+1) - 6n(n+1) + 6n) \]

\[ = \frac{1}{6n^2} (4n^3 - 3n^2 - 7n) \]

\[ = \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2} \rightarrow \frac{2}{3} \text{ as } n \rightarrow \infty \]

Thus the upper and lower limits equal \( \frac{2}{3} \) and this is the value of the integral. In the formal theory of Riemann integration it is proved in a general context why this is the case.

### A hint of Lebesgue measure theory

Modern integration theory is ultimately founded on Lebesgue (pronounced “le-baig”) measure theory. Lebesgue integration underpins modern probability theory, Fourier Theory, financial mathematics, etc. The motivation and development for measure theory (and Lebesgue’s version of it) is beyond this chapter but I can give a practical glimpse of how it can be used to integrate the problem described above. Very crudely instead of partitioning the domain of the function to build up little rectangles which you add up, you partition the range and then you work with the size (i.e. the measure) of the set of points in the domain which map to the relevant part of the range.

The following diagram sets out the process:
The recipe:

We partition $[0,1]$ into sets $A_i = [a_{i-1}, a_i]$ and we form upper and lower sums as follows:

- $U(n) = \sum_{i=1}^{n} a_i m(f^{-1}(A_i))$ and
- $L(n) = \sum_{i=1}^{n} a_{i-1} m(f^{-1}(A_i))$

What this recipe says is this. Take the partition $A_i$ and find those elements of the domain that map to it. Then find the measure of that set. The measure is in essence the length of the relevant interval/set that is established. Remember that $f^{-1}(A_i) = \{ x : f(x) \in A_i \}$. Measure theory generalises these concepts. So let’s do the upper and lower sums.

Our partition is $A_i = \left[ \frac{i-1}{n}, \frac{i}{n} \right]$ for $1 \leq i \leq n$. Note that $\sqrt{x_i} = a_i$ implies that $x_i = a_i^2$ so that $A_i = \{ x : a_{i-1}^2 \leq x \leq a_i^2 \} = \{ x : \left( \frac{i-1}{n} \right)^2 \leq x \leq \left( \frac{i}{n} \right)^2 \}$

The length (measure) of $A_i$ is then $\left( \frac{i}{n} \right)^2 - \left( \frac{i-1}{n} \right)^2$

Hence $U(n) = \sum_{i=1}^{n} \frac{i}{n} \left( \left( \frac{i}{n} \right)^2 - \left( \frac{i-1}{n} \right)^2 \right) = \sum_{i=1}^{n} \frac{2i^2 - 3i + 1}{n^3}$ which is what we got before.

Similarly $L(n) = \sum_{i=1}^{n} \frac{i-1}{n} \left( \left( \frac{i}{n} \right)^2 - \left( \frac{i-1}{n} \right)^2 \right) = \sum_{i=1}^{n} \frac{2i^2 - 3i + 1}{n^3}$ as before.

Lebesgue measure theory is fundamental to modern probability and Fourier theory. The high level motivation for Lebesgue integration is given in the main work.

5. Prove by induction for all positive integers:

$P(n) = x^n - y^n = (x - y) (x^{n-1} + x^{n-2} y + x^{n-3} y^2 + ... + xy^{n-2} + y^{n-1})$

Solution:

Let’s start with the basis step ie $n = 1$
LHS = $x - y$
RHS = $x - y$.

Note that when $n = 1$ the reducible second term is simply 1 ie it reduces to 1. We could simply use $n = 2$ as our more informative basis step, having proved that $n = 1$ works trivially. For $n = 2$ the RHS would be $(x - y) (x + y) = x^2 - y^2 = \text{LHS}$.

Suppose $P(n)$ is true for any $n$, then:

$x^{n+1} - y^{n+1} = x^{n+1} - x^n y + x^n y - y^{n+1} = x^n (x - y) + y (x^2 - y^2)$

$= x^n (x - y) + y(x - y) (x^{n-1} + x^{n-2} y + x^{n-3} y^2 + ... + xy^{n-2} + y^{n-1})$ using the induction step

$= (x - y) [ x^n + y(x^{n-1} + x^{n-2} y + x^{n-3} y^2 + ... + xy^{n-2} + y^{n-1}) ]$

$= (x - y) (x^n + yx^{n-1} + x^{n-2} y^2 + x^{n-3} y^3 + ... + xy^{n-2} + y^{n-1})$}

$= P(n + 1)$

This is what we wanted to prove. Note that starting at $n = 1$ is not particularly illuminating in terms of how one might get to the next stage. There are many cases where the basis step is so trivially obvious that it gives no hint as to how one might proceed. You will see further examples of this in the problem.
6. Doing it both ways - a note on combinatorial identities

There are essentially two ways to prove combinatorial identities: by combinatorial reasoning or purely algebraic processes. As an example consider the following identity: \(^nC_r = \binom{n}{n-r}\).

The combinatorial proof would go this way: the LHS is the number of ways of choosing \(r\) objects out of \(n\) while the RHS is the number of ways of not choosing \(n-r\) objects from \(n\). Picking a team of \(r\) players from \(n\) is the same as choosing the \(n-r\) to leave on the bench.

The algebraic proof follows from the definition of the symbol \(^nC_r = \frac{n!}{(n-r)!r!}\), and hence \(^nC_{n-r} = \frac{n!}{(n-(n-r))!(n-r)!} = \frac{n!}{r!(n-r)!}\).

Another example is \(r^nC_r = n^{n-1}C_{r-1}\)

The combinatorial proof runs like this: suppose we choose a team of \(r\) players from \(n\) (there are \(n\) ways of doing this) and for each of those teams we anoint one of those \(r\) players as captain. The LHS is the resulting number of combinations. Alternatively we could choose the captain first and there are \(n\) ways of doing that and, having done so, we have to choose \(r-1\) players from \(n-1\) so the number of combinations is \(n^{n-1}C_{r-1}\). Note that we implicitly rely upon the principle that if you count the same set in two different ways the answer is the same.

The algebraic proof is as follows: LHS = \(\frac{n!(n-1)!}{(n-r)!(r-1)!} = \frac{n(n-1)!}{(n-r)!(r-1)!}\).

RHS = \(\frac{n!}{(n-1)!(r-1)!} = \frac{n(n-1)!}{(n-1)!(r-1)!} = \) LHS

Problem

Using combinatorial and algebraic reasoning, prove that the number of distinguishable distributions when \(r\) indistinguishable balls are placed in \(n\) cells is:

\[D(r,n) = \binom{n+r-1}{r}\]

By way of example if 2 indistinguishable balls were placed in 3 cells the cell occupancy numbers would be distributed as follows, giving rise to 6 distinguishable distributions:

Cell 1 Cell 2 Cell 3
2 0 0
0 2 0
0 0 2
1 1 0
1 0 1
0 1 1

Solution:

The combinatorial reasoning runs like this. Symbolise a ball by a star and a cell is represented by two vertical bars | |. Thus \(n\) cells are represented by the spaces between \(n+1\) vertical bars. In the example above the last distribution would be represented by **|***

Because any distribution must start and end with a bar you are left with \(n-1\) bars and \(r\) stars to arrange in any order - a total of \(n+r-1\) symbols. Thus the number of distinguishable distributions arising from \(r\) indistinguishable balls being placed in the \(n\) cells is equivalent to choosing \(r\) out of the \(n+r-1\) objects (symbols). That number is \(\binom{n+r-1}{r}\). In deriving that result we focused on arranging the stars among the bars but one could symmetrically focus on arranging the \(n-1\) bars (being the ones that create the cells) among the \(n+r-1\) symbols and that number is \(\binom{n+r-1}{n-1}\) which must equal \(\binom{n+r-1}{r}\).

The algebraic proof simply relies upon properties of the combinatorial symbol. Thus \(\binom{n+r-1}{r} = \frac{(n+r-1)!}{(n-1)!r!}\) and \(\binom{n+r-1}{n-1} = \frac{(n+r-1)!}{r!(n-1)!}\) which establishes the result.

7. Pascal’s Identity

One of the most basic elements of combinatorial theory is \(^nC_r = \binom{n}{r}\), where \(0 \leq r \leq n\) and \(^nC_0 = 1\) and \(^nC_n = 1\).
Prove by induction $^nC_r = ^{n-1}C_{r-1} + ^{n-1}C_r$ where $1 < r < n$ (both integers) This is Pascal's Identity. Pascal's Triangle gives you the way to remember this formula:

1 1 1 1 1
1 2 1 1 6 4 1
1 3 1 1 1
1 5 10 10 5 1
1 7 21 35 35 21 7
1 9 36 84 126 126 84 36
1 11 55 220 495 792 1287 1716 1716 1287 792 495 220 55 11 1
1 13 78 455 1287 2520 4368 6188 6188 4368 2520 1287 455 78 13 1

etc

You should provide a combinatorial argument for the proposition that $^nC_r = ^{n-1}C_{r-1} + ^{n-1}C_r$. If you can do that you can provide an alternative proof for the binomial theorem which is the next problem.

Solution:

Here is a combinatorial argument for the proposition that $^nC_r = ^{n-1}C_{r-1} + ^{n-1}C_r$.

One way to prove this is to suppose that in a class of $n$ students one is "distinguished" in some way (I will leave it to your fertile imaginations to work out an appropriate distinguishing characteristic) and out of these $n$ students we want to choose a group of $r$. We really have only 2 mutually exclusive possibilities with this "distinguished" student: we can include or exclude him/her. That's significant because we know that the number of objects in the union of two sets $A$ and $B$ is: $N(A \cup B) = N(A) + N(B)$ when $A \cap B = \emptyset$, i.e., the sets are disjoint. Let's suppose we include the student (this is set $A$). Then we can choose the remaining $r - 1$ students from $n - 1$ in $^{n-1}C_{r-1}$ ways.

Alternatively, if we exclude the student (this is set $B$) we have to choose the whole group of $r$ from $(n - 1)$ individuals and that is simply $^{n-1}C_r$ or $N(B)$.

So when we choose $r$ from $n$ which is $^nC_r$ or $N(A \cup B)$ (ie choosing $r$ from $A \cup B$) we get $^{n-1}C_{r-1} + ^{n-1}C_r$. That's where the formula comes from.

You have to prove $P(n) = ^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$ by induction.

Basis step: $P(2) = ^{n-1}C_1 + ^{n-1}C_2$ and we know that this is the number of ways of choosing 2 objects (unordered) from $n$ distinguishable objects

Using the logic of inclusion - exclusion developed above we see that:

The number of ways of choosing 2 objects (unordered) from $n$ distinguishable objects $= [\text{the number of ways of choosing 1 object from } n-1 \text{ distinguishable objects}] + [\text{the number of ways of choosing 2 objects (unordered) from } n-1 \text{ distinguishable objects}]

$= ^{n-1}C_1 + ^{n-1}C_2

= \frac{(n-1)!}{(n-2)! \cdot 1!} + \frac{(n-1)!}{(n-3)! \cdot 2!}

= \frac{(n-1)!}{(n-3)!} \left( \frac{1}{n-2} + \frac{1}{2} \right)

= \frac{n(n-1)!}{(n-2)! \cdot 2!}

= \frac{n(n-1)}{2!}

So we have established that $P(2)$ is true. We could have started with $n=1$ in which case there are $n=C_1$ ways of choosing $n$ distinguishable objects but that doesn't really illuminate how the more general proof would proceed.

Induction step: Assume $P(n)$ is true for all $r \leq n$. Note that we have to pick up not just $r$ but $r-1$ so we frame our induction in this way (ie strong induction).

$P(n+1) = ^nC_{r-1} + ^nC_r

= \frac{n(n-1)(n-2)\ldots(n-r+2)}{(r-1)!} + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}

= \frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{r!} \left( 1 + \frac{r}{n-r+1} \right)

= \frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{r!} \left( \frac{n-r+1}{n-r+1} \right)
\[
= \frac{(n+1)(n-1)(n-2)...(n+1-r-1)}{r!} \\
= n+1 \binom{n}{r}
\]
So the formula is true for \( n+1 \) and the principle of (strong) induction establishes that it is true for all \( n \geq 2 \)

8. The binomial theorem

This is without doubt one of the most ubiquitous formulas in mathematics. There are many fundamental problems in probability theory which turn upon the binomial theorem.

(i) You have to prove \( P(n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x + y)^n \) is true for all integers \( n \geq 1 \).

(ii) As an application prove by induction that \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} C^2_k = C^0_n - C^2_n + C^4_n - ... + (-1)^n C^2_n = 0 \)

(iii) Conjecture what the following sum is and then prove your conjecture by induction:

\[ 0^n C_0 + 1^n C_1 + 2^n C_2 + ... + (n-1)^n C_{n-1} + n^n C_n \]

Try to provide a combinatorial derivation for your result.

(iv) Vandermonde’s Theorem

First a bit of notation: if \( m \) is any real number and \( n \) is a positive integer then \( m_n = m (m - 1) (m - 2) ... (m - n + 1) \). Thus \( m_n = n! \binom{m}{n} \)

Vandermonde’s Theorem asserts that for any two real numbers \( a \) and \( b \), and any positive integer \( n \) the following holds:

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
\]

This looks like the Binomial Theorem so it should be amenable to an inductive proof. There are two hints:

(a) Work out what you have to multiply \( (a + b)^n \) by to get \((a + b)^{n+1}\)

(b) Whatever you multiply the LHS by you also have to multiply each term on the RHS by. At this stage you have to be a bit cunning and break up the number in a way which will allow you to use Pascal’s Identity.

Solution:

Let’s prove the formula in a way which involves use of sigma and combinatorial notation. This will give you practice in manipulating symbols. You may think that this is useless. Wrong! Tensor calculus is essentially based on symbolic manipulation at a purely calculational level. If you ever study general relativity you will appreciate this point. If you are still not convinced, a 1965 Nobel Prize winner in physics, Julian Schwinger, has written a book on quantum theory which is based entirely on abstract symbols (“Quantum Mechanics: Symbolism of Atomic Measurements”, Springer, 2001). In effect he abstracts the physical logic of quantum phenomena. Symbols do matter!

Let \( P(n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x + y)^n \). We can also write this as \( P(n) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x + y)^n \) by symmetry.

This is the binomial formula. Look at the structure of the formula on the left. You may recognise it better this way:

\[
P(n) = C^n_0 x^n y^0 + C^n_1 x^{n-1} y^1 + C^n_2 x^{n-2} y^2 + ... + C^n_{n-1} x^1 y^{n-1} + C^n_n x^0 y^n
\]

\[= x^n + C^n_1 x^{n-1} y + C^n_2 x^{n-2} y^2 + ... + C^n_{n-1} x y^{n-1} + y^n
\]

We want to prove that \( P(n+1) = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k \) by the principle of induction. Clearly \( P(1) \) is true.

\[\sum_{k=0}^{n} C^k_1 x^{n+1-k} y^k = C^n_0 x^{n+1-0} y^0 + C^n_1 x^{n+1-1} y = x + y = (x + y)^1
\]

Now \( P(n+1) = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k \)

\[= \sum_{k=0}^{n} C^n_k (x^{n+1-k} y^k + x^{n-k} y^{k+1})
\]

\[= \sum_{k=0}^{n} C^n_k x^{n+1-k} y^k + \sum_{k=0}^{n} C^n_k x^{n-k} y^{k+1}
\]

\[= A + B \quad (1)
\]

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Now we manipulate (1) bearing in mind we want to get to this:  \( P(n+1) = \sum_{i=0}^{n+1} C_i^0 x^{n+1-i} y^i \)

Along the way we will need to use the fact that \( a^{n+1} C_i = a C_{i-1} + a C_i \) which has been proved already (Pascal’s Identity – Problem 7)

Taking the two components of (1) separately we get:

\[ A = \sum_{i=0}^{n+1} C_i^0 x^{n+1-i} y^i = \sum_{i=0}^{n+1} C_i^0 x^{n+1-i} y^i \]. Why is this true? We have added an extra term and the only way the statement would be true is if the extra term is 0. Why is this true? Because \( C_r^0 = 0 \) for \( r > n \) since there are 0 ways of drawing \( r \) objects from \( n \) when \( r > n \). Hence \( C_{n+1}^0 = 0 \).

The second part is \( B = \sum_{i=0}^{n+1} C_i^0 x^{n+1-i} y^i \). Guided by our already proved combinatorial result we try the following:

\[ B = \sum_{i=0}^{n+1} C_i^0 x^{n+1-i} y^i = \sum_{i=0}^{n+1} C_i^{n+1} x^{n+1-i} y^i \]. Why did I do this and is it correct?

Basically we want to get the following : \( \sum_{i=0}^{n+1} (C_i^0 + C_i^{n+1}) x^{n+1-i} y^i \) because we can then use Pascal’s identity to show that

\[ \sum_{i=0}^{n+1} (C_i^0 + C_i^{n+1}) x^{n+1-i} y^i = \sum_{i=0}^{n+1} C_i^{n+1} x^{n+1-i} y^i \]

So all we have to do is show is that \( B = \sum_{i=0}^{n+1} C_i^0 x^{n+1-i} y^i = \sum_{i=0}^{n+1} C_i^{n+1} x^{n+1-i} y^i \) and we are home. Let \( j = i - 1 \)

Then \( B = \sum_{i=0}^{n+1} C_i^0 x^{n+1-i} y^i = \sum_{i=0}^{n+1} C_i^{n+1} x^{n+1-(i+1)} y^{i+1} \)

\[ = \sum_{i=0}^{n+1} C_i^{n+1} x^{n-i} y^{i+1} \]

But what does it mean for \( j \) to run from -1 to \( n \) ? As a matter of logic and convention \( C^n_{-1} = 0 \) when \( j < 0 \) so we can run \( j \) from 0 to \( n \). In other words:

\[ B = \sum_{j=0}^{n} C_j^n x^{n-j} y^{j+1} = \sum_{j=0}^{n} C_j^n x^{n-j} y^{j+1} = \sum_{j=0}^{n+1} C_j^n x^{n+1-j} y^j \] from (2). Note that \( j \) and \( i \) are merely placeholders or dummy variables so, for instance:

\[ \sum_{i=0}^{n+1} C_i^n x^{n-i} y^{i+1} = \sum_{i=0}^{n+1} C_i^n x^{n-i} y^{i+1} \]

When we put the two components of (1) together we get:

\[ A + B = \sum_{i=0}^{n+1} C_i^0 x^{n+1-i} y^i + \sum_{i=0}^{n+1} C_i^{n+1} x^{n+1-i} y^i = \sum_{i=0}^{n+1} (C_i^0 + C_i^{n+1}) x^{n+1-i} y^i = \sum_{i=0}^{n+1} C_i^{n+1} x^{n+1-i} y^i \]

So we see that \( P(n+1) \) is true. You can get to the same result by multiplying \( (\sum_{i=0}^{n} C_i^n x^{n-i} y^i) (x + y) \) out and combining pairs which involves Pascal’s Identity.

Technically you should demonstrate that you can match pairs appropriately which actually amounts to what has been done above.

The only other thing you have to watch is the two end points where you need to note that \( C_0^0 = C_0^{n+1} \) and \( C_n^0 = C_n^{n+1} \)

The ability to manipulate sigma notation can prove very useful in certain problems involving multiplication of matrices.

**Historical note:**

Newton asserted that for any real number \( a \) and any \( x \) such that \( |x| < 1 \) the following holds:

\[ (1 + x)^n = 1 + ax + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^3 + ... \]

Thus for non-integral exponents an infinite series arises. It was Cauchy who supplied a more rigorous proof of the binomial theorem in all its generality. You can read for yourself how Newton guessed the result by going to this link: http://www.macaulester.edu/arata/edition2/chapter2/chapt2d.pdf

If you want to see how the theorem is proved for any rational exponent you need to use Taylor’s Theorem with the Cauchy form of the remainder. Those who feel a bit frisky can go to Problem 49 where all is revealed. It takes a little more work to establish the binomial theorem for any real exponent.

**Alternative proof:**

If you don't like that proof maybe this one will appeal.
\((x + y)^n\) is a polynomial of degree \(n\) in \(x\) and \(y\). This means that the indices of \(x\) and \(y\) add to \(n\) eg \(x^{n-2}y^2\). To find the expansion of 
\((x + y)^n\) you systematically multiply out the \(n\) factors and this involves choosing \(x\) and \(y\) in every possible way, multiplying the chosen terms and then adding. A general term \(x^{n-k}y^k\) is found by choosing \(y\) from \(k\) factors and \(x\) from \(n-k\) factors. This can be done in \(\binom{n}{k}\) ways so that is the coefficient of \(x^{n-k}y^k\) and you add up all such terms from 0 to \(n\).

This type of reasoning underpins techniques used in calculating probabilities using probability generating functions. For instance, the probability generating function (pgf) of the score obtained with one throw of an unbiased die is 
\[
\frac{1}{6}(1 + x + x^2 + x^3 + x^4 + x^5)
\]
and for \(n\) throws the pgf is \(\left(\frac{1}{6}\right)(1 + x + x^2 + x^3 + x^4 + x^5)^n\). To find the probability of obtaining a total of 10 in 5 throws of an unbiased die you need to find the coefficient of \(x^{10}\) in \(\left(\frac{1}{6}\right)(1 + x + x^2 + x^3 + x^4 + x^5)^5\). The answer is \(\frac{126}{4^{10}} = \frac{7}{432}\).

The details are best left to a more detailed course on probability theory.

(ii) As an application prove by induction that 
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = C_n^0 - C_n^1 + C_n^2 - \ldots + (-1)^n C_n^n = 0
\]

**Solution:**

The easy way is to put \(x = 1\) and \(y = -1\) in the binomial formula and the result is immediate. It is much more tedious and error prone to go through a mechanical induction proof but here it is anyway.

The formula is true for \(n=1\) since 
\[
\sum_{k=0}^{1} (-1)^k \binom{1}{k} = 1 - 1 = 0.
\]

Assume the formula is true for any positive integer \(n\) ie \(\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0\). Then \(\sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} = \sum_{k=0}^{n} (-1)^k \left[ \binom{n}{k} + \binom{n}{k-1} \right]\) using Pascal's Identity

\[
eq \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} + \sum_{k=0}^{n+1} (-1)^k \binom{n}{k-1}
\]

\[
eq \sum_{k=0}^{n+1} (-1)^k \binom{n}{k} + \sum_{k=1}^{n+1} (-1)^k \binom{n}{k-1}
\]

because \(\binom{n}{n+1} = 0\)

\[
eq 0 + \sum_{k=1}^{n+1} (-1)^k \binom{n}{k-1}
\]

using the induction hypothesis and the fact that \(\binom{n}{r} = 0\) when \(r < 0\) (this way we get a consistent combinatorial meaning - you can’t choose a negative number of objects)

\[
eq \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j-1} \text{ by letting } j = k - 1
\]

\[
eq - \sum_{j=0}^{n-1} (-1)^{j+1} \binom{n}{j-1} \text{ again using the fact that } \binom{n}{n-1} = 0
\]

\[
eq 0 \text{ (again using the induction hypothesis - note that we have assumed that the formula } \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \text{ holds for any } n \text{ because in the final step we re asserting the induction hypothesis is true for } n-1.

As you can see - the first proof is preferable !

(iii) Conjecture what the following sum is and then prove your conjecture by induction:

\[
0 \binom{n}{0} + 1 \binom{n}{1} + 2 \binom{n}{2} + \ldots + n \binom{n}{n}
\]

Try to provide a combinatorial derivation for your result.

**Solution**

The \(\sum_{k=0}^{n} k \binom{n}{k}\) and this suggests that an approach based on taking derivatives might be fruitful because 
\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k
\]
by the Binomial Theorem and when we differentiate each side we get
\[
n(1 + x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1}.
\]

When this is evaluated at \(x = 1\) we get as our conjecture: 
\[
n2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}
\]
The inductive proof of this proceeds by first establishing that the formula is correct for $n = 1$.
For $n = 1$ the LHS $= 1$ and the RHS $= 1.C_1^1 = 1$

Our inductive hypothesis is that $n2^{n-1} = \sum_{k=0}^{n} k C_{k}^{n}$

Now $\sum_{k=0}^{n+1} k C_{k}^{n+1} = \sum_{k=0}^{n} k ( C_{k}^{n} + C_{k-1}^{n} ) + n + 1$ using Pascal’ s Identity

= $n2^{n-1} + n + 1 + C_{k-1}^{n} + \sum_{k=1}^{n} k C_{k-1}^{n}$ using the induction hypothesis

= $n2^{n-1} + n + 1 + \sum_{k=0}^{n} (j+1) C_{j}^{n}$ by putting $j = k - 1$

= $n2^{n-1} + n + 1 + \sum_{j=0}^{n} C_{j}^{n} - n + \sum_{j=0}^{n} C_{j}^{n} - 1$

= $n2^{n-1} + n2^{n-1} + 2^{n}$ using the induction hypothesis again and the fact that $\sum_{j=0}^{n} C_{j}^{n} = 2^{n} = \sum_{j=0}^{n-1} C_{j}^{n} + 1$

= $n2^{n-1} + 2^{n}$

= $(n + 1)2^{n}$ which establishes that the formula is true for $n + 1$ and hence true for all $n$ by induction.

One approach to a combinatorial proof is to view the sum $0 C_{0}^{n} + 1 C_{1}^{n} + 2 C_{2}^{n} + \ldots + n C_{n}^{n}$ as the weighted sum of all subsets of size $k$ ($0 \leq k \leq n$) taken from the set of $n$ elements. There are $2^{n}$ subsets that can be obtained from a set of $n$ elements. There are $n + 1$ "weights" ie 0, 1, 2, ..., $n$ associated with each subset of size $k$. Note that 0 is a legitimate weight. The sum of these weights is:

$$0 + 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$ Hence the average weight is: $\frac{\frac{n(n+1)}{2}}{2^{n}} = \frac{n}{2}$

So we have $2^{n}$ subsets with an average weight of $\frac{n}{2}$ so the total weight is $\frac{n}{2}2^{n} = n2^{n-1}$ and this equals $0 C_{0}^{n} + 1 C_{1}^{n} + 2 C_{2}^{n} + \ldots + n C_{n}^{n}$

This was a problem in I. Lovász, J Pelikán and K Vesztergombi, *Mathematics: Elementary and Beyond*, Springer, 2003, page 19. The authors don’t give a hint or solution for this problem, which is rather surprising since the book is aimed at students.

(iv) Vandermonde’s Theorem

Let the theorem be denoted by $T_{n}$. Now asserts $T_{n}$ that $(a + b)_1 = 1. a_1 + 1. b_1$ and this is true since both sides do equal $a + b$. As our induction hypothesis assume that $(a + b)_k = \begin{pmatrix} k \\ 0 \end{pmatrix} a_1 + \begin{pmatrix} k \\ 1 \end{pmatrix} a_{k-1} b_1 + \begin{pmatrix} k \\ 2 \end{pmatrix} a_{k-2} b_2 + \ldots + \begin{pmatrix} k \\ r \end{pmatrix} a_{k-r} b_r + \ldots + \begin{pmatrix} k \\ k \end{pmatrix} b_k$ for any $k$.

We are after an expression for:

$$(a + b)_{k+1} = (a + b) (a + b - 1) \ldots (a + b - (k + 1) + 1) = (a + b) (a + b - 1) \ldots (a + b - k + 1) (a + b - k) = (a + b)_k (a + b - k)$$

So our multiplying factor is $(a + b - k)$. At this stage you might want to consider hint (b) and the reference to Pascal’ s Identity which is:

$$\begin{pmatrix} n \\ r - 1 \end{pmatrix} + \begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n + 1 \\ r \end{pmatrix}$$ where $n$ and $r$ are positive integers such that $n >= r >= 1$

On the RHS the multiplication proceeds as follows:

$$\begin{pmatrix} k \\ 0 \end{pmatrix} a_k \{ a - k + b \}$$

$$\begin{pmatrix} k \\ 1 \end{pmatrix} a_{k-1} b_1 \{ a - k + 1 \} + (b - 1)$$

$$\begin{pmatrix} k \\ 2 \end{pmatrix} a_{k-2} b_2 \{ a - k + 2 \} + (b - 2)$$

$$\ldots$$

$$\begin{pmatrix} k \\ r \end{pmatrix} a_{k-r} b_r \{ a - k + r \} + (b - r)$$

$$\ldots$$

$$\begin{pmatrix} k \\ k \end{pmatrix} b_k \{ a + (b - k) \}$$

Note here that $(a - k + 1) + (b - 1) = a + b - k$ but it has been broken up judiciously into two components while not changing the overall sum. The first component is $a - (k - s)$ where $s$ runs from 0 to $k$. The second component is $(b - r)$ where $r$ symmetrically runs from 0 to $k$. Thus everything on the LHS and RHS has been multiplied by $a - k + b$. 

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Now $a_k \{ a - k + b \} = a(a-1)...(a-k+1) (a-k) + b_1 a_k = a_k + b_1$ (note that $b = b_1$ and the sum of the subscripts is $k+1$)

$$a_{k-1} b_1 \{ (a - k + 1) + (b - 1) \} = a_{k-1} (a - k + 1) b_1 + a_{k-1} b_1 (b - 1) = a (a-1)...(a-k+2)(a-k+1) b_1 + a_{k-1} b_1 = a_k b_1 + a_{k-1} b_1$$

$$a_{k-2} b_2 \{ (a - k + 2) + (b - 2) \} = a(a-1)...(a-k+3) (a-k+2) b_2 + b_2 (b-2) = a_{k-1} b_2 + a_{k-2} b_1$$

More generally,

$$a_{k+r} b_r \{ (a - k + r) + (b - r) \} = a(a-1)...(a-k+r+1) (a-k+r) b_r + a_{k+r} b_r = a_{k+1-r} b_r + a_{k-r} b_{r+1}$$

Thus the final term is $b_k \{ a + (b - k) \} = b_k a_1 + b_k (b - k) = b_k a_1 + b_{k+1}$.

Now putting this all together with the combinatorial factors and pairing appropriately we get:

$$\{ 0 \} a_{k+1} + a_k b_1 + \{ 1 \} a_0 b_1 + a_{k-1} b_2 + \{ 2 \} a_{k-1} b_2 + a_{k-2} b_3 + ... + \{ k \} a_{k+r} b_{r-1} + a_{k+r} b_r = \{ 0 \} a_{k+1} + a_k b_1 + \{ 1 \} a_0 b_1 + a_{k-1} b_2 + ... + \{ k \} a_{k+r} b_{r-1} + a_{k+r} b_r = \{ k+1 \} a_{k+1} + \{ k+1 \} a_{k+1} b_1 + \{ k+1 \} a_{k-1} b_2 + ... + \{ k+1 \} a_{k+r} b_{r-1} + a_{k+r} b_r.$$ 

using Pascal's Identity in each of the curly brackets and also noting that $\{ 0 \} a_{k+1} = \{ k+1 \} a_{k+1}$ and $\{ k \} b_k = \{ k+1 \} b_k$.

Thus it has been established that $\sum_{i=1}^{k+1} a_i$ is true and so Vandermonde's Theorem is true by induction.

9. An inequality involving $e$

Prove by induction: $n! > \left( \frac{n}{e} \right)^n$ $\forall n \geq 1$ $\in N$

You can assume that $1 + \frac{1}{n} < e$ $\forall n$ $1$ but you should be able to see why.

Solution

Since $1 + \frac{1}{n}$ is an increasing sequence bounded above by we have that $1 + \frac{1}{n} < e$ $\forall n \geq 1$ (see the reference below to see why this is so). The issue of equality is one of those minor points you actually need to worry about to get a rigorous proof of the proposition. If it were true that $\exists n$ such that $1 + \frac{1}{n} = e$ you would get a contradiction since $e$ is the least upper bound of the infinite sequence $\{ n \in N : 1 + \frac{1}{n} \}$. Remember that an upper bound is a number which dominates every member of the sequence. A least upper bound is the smallest of the upper bounds. Thus, as an example, 10 is an upper bound of $1 + \frac{1}{n}$ for $n \geq 1$ but so are 9, 8, 7 etc. In fact 2 is the least upper bound since $1 + \frac{1}{n} < 2$ for all $n \geq 1$.

This means that since $e$ is a least upper bound of the sequence $1 + \frac{1}{n}$, it follows that $1 + \frac{1}{n} < e$ $\forall n \geq 1$. Let's assume equality for some $n$. But we know that the sequence is strictly increasing ie $\left( 1 + \frac{1}{n+1} \right)^{n+1} > \left( 1 + \frac{1}{n} \right)^n = e$ (see All you wanted to know about $e$ but were afraid to ask! in the main work). In other words you would have an infinite number of members of the sequence greater than $e$ which is supposed to dominate them! It would also mean that $e$ is not transcendental.

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transcendental in 1873.

So I think we can take it that \( (1 + \frac{1}{n})^n < e \forall n \geq 1 \).

The formula is true for \( n = 1 \) since \( 1 > \left( \frac{1}{e} \right)^1 \)

Assume the formula is true for any \( n \) and consider \( (n+1)! \).

We know that \( e > (1 + \frac{1}{n})^n \)

So \( (n+1)! > (n+1)! \left( \frac{1}{n} \right) = (n+1)! \left( \frac{n+1}{n} \right) n! \left( \frac{n+1}{n} \right) = \frac{n!}{e} \)

Therefore \( (n+1)! > \frac{(n+1)\pi^2}{e} \) i.e. \( (n+1)! > \left( \frac{n+1}{e} \right)^n \)

In other words the formula is true for \( n+1 \) and hence true for all \( n \) by the principle of induction.

**Alternative solution not involving induction:**

You know that \( e^n = 1 + n + \frac{n^2}{2!} + \frac{n^3}{3!} + \ldots \ldots + \frac{n^n}{n!} \)

The inequality can be rewritten as \( n! \cdot e^n > n^n \)

The LHS of this inequality is \( n! \cdot (1 + \frac{n^2}{2!} + \frac{n^3}{3!} + \ldots \ldots + \frac{n^n}{n!}) \)

\( (n+1)! > \frac{(n+1)^{n+1}}{e} \) which might cause Monsieur Hermite to turn in his grave since he proved (because \( e \) would equal a finite binomial series) which might cause Monsieur Hermite to turn in his grave since he proved \( e \) was transcendental in 1873.

**10. An exploration of \( \pi \)**

In the 17th century an English mathematician John Wallis did some fundamental exploration into \( \pi \). Wallis' formula for \( \pi \) is:

\[
\pi = 2 \left( \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots \ldots \right)
\]

The product of the \( k \)th pair of fractions in \( \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots \ldots \) is \( \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} = \frac{4k^2}{4k^2-1} \)

It then follows that \( \pi = 2 \left[ \prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} \right] \). This converges slowly. For instance for \( k = 500, \pi \) is only correct to 2 decimal places.

Wallis was a class act. Newton carefully studied Wallis' books. To get his formula he played around with integrals of the form:

\[
\int_0^1 \left( 1 - x^p \right)^q \text{d}x
\]

He used the binomial theorem to expand the integrand for \( q \) positive and integral (and \( p \) could be anything). The challenge is to do that for \( q = 0,1,2 \) and 3 and then guess the form of \( q=4 \). Wallis got the following formula for \( q \) in general:

\[
f(p,q) = \frac{1}{\int_0^1 \left( 1 - x^p \right)^q \text{d}x} = \frac{(p+1)(p+2)(p+3)\ldots(p+q)}{q!}
\]

I invite you to prove that \( \int_0^1 \left( 1 - x^p \right)^q \text{d}x = \frac{q}{p+q} \int_0^1 \left( 1 - x^p \right)^{q-1} \text{d}x \) and then use induction on \( q \) to prove that \( f(p,q) = \frac{(p+1)(p+2)(p+3)\ldots(p+q)}{q!} \)

Wallis basically established that \( f(\frac{1}{2},\frac{1}{2}) = \frac{4}{\pi} \). Now \( \int_0^1 \left( 1 - x^2 \right)^{\frac{1}{2}} \text{d}x \) is simply the area of a quadrant of a circle radius 1 and that area is \( \frac{\pi}{4} \).

**Solution**

\[
\int_0^1 \left( 1 - x^2 \right)^{\frac{1}{2}} \text{d}x = \int_0^1 \text{d}x = 1
\]
\[ \int_0^1 \left( 1 - x^p \right)^3 \, dx = \left[ x - \frac{x^{1/p}}{1 + \frac{1}{p}} \right]_0^1 = 1 - \frac{p}{p+1} = \frac{1}{p+1} \]

\[ \int_0^1 \left( 1 - x^p \right)^2 \, dx = \int_0^1 \left( 1 - 2x^p + x^{2p} \right) \, dx = \left[ x - 2 \frac{x^{1/p}}{1 + \frac{1}{p}} + \frac{x^{1/p}}{1 + \frac{1}{p}} \right]_0^1 = \frac{2}{p+1} \]

\[ \int_0^1 \left( 1 - x^p \right)^3 \, dx = \int_0^1 \left( 1 - 3x^p + 3x^{2p} - x^{3p} \right) \, dx = \left[ x - 3 \frac{x^{1/p}}{1 + \frac{1}{p}} + \frac{3x^{1/p}}{1 + \frac{1}{p}} - \frac{x^{1/p}}{1 + \frac{1}{p}} \right]_0^1 = 1 - \frac{3p}{p+1} + \frac{3p}{p+2} - \frac{p}{p+3} \]

Hence

\[ \int_0^1 \left( 1 - x^p \right)^4 \, dx = \frac{4!}{(p+1)(p+2)(p+3)(p+4)} \]

The next insight is to note that

\[ C_q^p = \frac{(p+q)!}{p!q!} \implies \frac{q!}{(p+1)(p+2)...(p+q)} = \frac{(p+q)!}{p!q!} \frac{1}{(p+1)(p+2)...(p+q)} = \frac{1}{C_q^p} \]

So Wallis worked with the reciprocal of the integrals ie.

\[ f(p, q) = \frac{1}{\int_0^1 \left( 1 - x^p \right)^q \, dx} = \frac{1}{\int_0^1 \left( 1 - x^p \right)^q \, dx} \]

\[ f(p, q) = \frac{1}{\int_0^1 \left( 1 - x^p \right)^q \, dx} = \frac{1}{\int_0^1 \left( 1 - x^p \right)^q \, dx} \]

As you can see, it's getting pretty messy so at this stage we search for the pattern. For \( q = 4 \) we would guess:

\[ \int_0^1 \left( 1 - x^p \right)^4 \, dx = \frac{4!}{(p+1)(p+2)(p+3)(p+4)} \]

The next insight is to note that \( C_q^p \) is

\[ C_q^p = \frac{(p+q)!}{p!q!} \implies \frac{q!}{(p+1)(p+2)...(p+q)} = \frac{(p+q)!}{p!q!} \frac{1}{(p+1)(p+2)...(p+q)} = \frac{1}{C_q^p} \]

Hence

\[ \frac{q!}{(p+1)(p+2)...(p+q)} = \frac{1}{C_q^p} \]

So Wallis worked with the reciprocal of the integrals ie. \( f(p, q) = \frac{1}{\int_0^1 \left( 1 - x^p \right)^q \, dx} = \frac{1}{\int_0^1 \left( 1 - x^p \right)^q \, dx} \) [A]

This is now classic case for using induction to demonstrate that [A] is valid. One's mathematical instincts would suggest that the guess is "on the money".

The starting point is to prove

\[ \int_0^1 \left( 1 - x^p \right)^q \, dx = \frac{q!}{p!q!} \int_0^1 \left( 1 - x^p \right)^{q-1} \, dx \]

and you have to use integration by parts.

Let \( I = \int_0^1 \left( 1 - x^p \right)^q \, dx \) and remember that \( \int u \, dv = uv - \int v \, du \). To evaluate I we take:

\[ dv = dx \Rightarrow v = x \] and

\[ u = \left( 1 - x^p \right)^q \Rightarrow du = q \left( 1 - x^p \right)^{q-1} \frac{1}{p} x^{1/p} \, dx \]

So \( I = \int_0^1 \left( 1 - x^p \right)^q \, dx = \frac{q}{p!} \int_0^1 \left( 1 - x^p \right)^{q-1} x^{1/p} \, dx \)

But \( I = \int_0^1 \left( 1 - x^p \right)^q \, dx = \int_0^1 \left( 1 - x^p \right)^{q-1} \left( 1 - x^p \right) \, dx = \int_0^1 \left( 1 - x^p \right)^{q-1} \, dx - \int_0^1 \left( 1 - x^p \right)^{q-1} x^{1/p} \, dx = I - \frac{q}{p} I \)

where \( J = \int_0^1 \left( 1 - x^p \right)^{q-1} \, dx \) and \( I = \int_0^1 \left( 1 - x^p \right)^{q-1} \frac{1}{x^{1/p}} \, dx \)
By decomposing \( \frac{p}{q} f(k) \), we have shown that

\[
1 = \frac{q}{p+q} \int_0^1 \left( 1 - x \right)^q d\ x
\]

Thus we have shown that \( \int_0^1 \left( 1 - x \right)^q d\ x = \frac{q}{p+q} \int_0^1 \left( 1 - x \right)^{(q-1)} d\ x \) which is a recursive relationship amenable to an inductive proof on \( q \).

We know that \( f(p,0) = \int_0^1 \left( 1 - x \right)^0 d\ x = \int_0^1 d\ x = 1 \). Note that \( f(p,q) = \frac{(p+q)!}{p!q!} \) so \( f(p,0) = \frac{p!}{p!} = 1 \) so our base case is established. Now assume that \( [A] \) holds for any positive and integral \( q-1 \). Thus our induction hypothesis is:

\[
f(p, q - 1) = \frac{(p+1)(p+2)\ldots(p+q-1)}{(q-1)!}
\]

So \( f(p, q) = \frac{1}{p+q} \int_0^1 \frac{1}{\left( 1 - x \right)^q} dx = \frac{(p+q)!}{p!q!} \frac{(p+1)(p+2)\ldots(p+q-1)}{(q-1)!} = \frac{(p+1)(p+2)\ldots(p+q)}{q!} \). Hence \( [A] \) is true for \( q \) and so the formula is true for all \( q \) by induction.


(b) In a recent article in The American Mathematical Monthly the proof of the following proposition is sketched:

\[
I = \int_0^1 \frac{x^4(1-x)^6}{1 + x^2} \ dx = \frac{22 \pi}{7} \quad \text{(see Stephen K Lucas, "Approximation to \pi \ derived from integral with nonnegative integrands", The American Mathematical Monthly, Vol 116, Number 2, February 2009, pages 166 -172 )}
\]

By decomposing \( \frac{x^4(1-x)^6}{1 + x^2} \) into partial fractions do a straightforward integration to establish the result. If you don't know what a partial fraction is consider this partial decomposition: \( \frac{2x-1}{x^2+1} = \frac{x}{x^2+1} - \frac{1}{x+1} \). Multiply the RHS out to convince yourself of the equality. Note that this decomposition has the form \( \frac{P(x)}{Q(x)} = \frac{R(x)}{x} \), where the degrees of \( P(x) \) and \( R(x) \) are less than the degrees of \( Q(x) \) and \( S(x) \) respectively. See the Appendix for more detail on partial fractions.

**Solution to (b)**

The first thing to note is that the maximum power in the numerator is 8 so if the decomposition is of the form \( \frac{b x + c}{1 + x^2} \) where \( P(x) \) is a polynomial of degree 6 since it will be multiplied by \( 1 + x^2 \). This means we are looking for something of the form \( \frac{x^4(1-x)^6}{1 + x^2} = P(x) + \frac{b x + c}{1 + x^2} \)

\[
= \sum_{i=0}^6 a_i x^i + \frac{b x + c}{1 + x^2}
\]

where the \( a_i \), \( b \) and \( c \) have to be determined

Now \( x^4(1 - x)^6 = (1 + x^2)^6 \sum_{i=0}^6 a_i x^i \) + \( bx + c \)

We expand the LHS and equate coefficients:

\[
x^4(1 - x)^6 = x^4(1 - 4x + 6x^2 - 4x^3 + x^4) \]

\[
= x^4 - 4x^5 + 6x^6 - 4x^7 + x^8
\]

\[
(1 + x^2)^6 \sum_{i=0}^6 a_i x^i + bx + c = \sum_{i=0}^6 a_i x^i + \sum_{i=0}^6 a_i x^{i+2} + bx + c
\]

\[
= (a_0 + c) + (a_1 + b) x + (a_0 + a_2) x^2 + (a_1 + a_3) x^3 + (a_2 + a_4) x^4 + (a_3 + a_5) x^5 + (a_4 + a_6) x^6 + a_6 x^7 + a_6 x^8
\]

Now we just have to systematically equate coefficients:

\[
\begin{align*}
    a_0 + c &= 0 \\
    a_1 + b &= 0 \\
    a_0 &= 0 \\
    a_2 &= 0 \\
    a_3 &= 0 \\
    a_4 &= 1 \\
    a_5 &= 0 \\
    a_6 &= 6
\end{align*}
\]
Thus we have that:

Now let \( j = k + 1 \)

So we assume that the formula holds for any \( n \).

The proposition holds for \( n = 1 \) since the LHS = \( \frac{x^4(1-x)^4}{1+x^2} \) and the RHS = \( (x^6 \cdot 4x^3 + 5x^4 \cdot 4x^2 + 4) \sum_{k=0}^{n} (-4)^{-k} x^4k(1-x)^4k + \frac{-4}{1+x^2} \)

(c) People have investigated families of integrals similar to the one in (b). The most obvious generalisation is: \( I_{2n} = \int_{0}^{\infty} \frac{x^{2n}(1-x)^{2n}}{1+x^2} \) dx.

According to Lucas in the article cited in (b) there is a "straightforward" proof by mathematical induction of the following equality:

\[
\frac{x^4(1-x)^4}{1+x^2} = (x^6 \cdot 4x^3 + 5x^4 \cdot 4x^2 + 4) \sum_{k=0}^{n} (-4)^{-k} x^4k(1-x)^4k + \frac{-4}{1+x^2}
\]

Provide the inductive proof.

It can be shown that \( \frac{(-1)^n}{2n+1} \int_{0}^{\infty} \frac{x^{2n}(1-x)^{2n}}{1+x^2} \) dx = \( \pi \cdot \frac{n(n+1)}{(2n+1)(2n+3)} \sum_{k=0}^{n} \frac{(-4)^{-k} x^4k(1-x)^4k + \frac{-4}{1+x^2}}{1+x^2} \) (820k^3 + 1533k^2 + 902k + 165). This only involves some integration and the fact that \( \int_{0}^{\infty} x^n(1-x)^m \) dx = \( \frac{m! n!}{(m+n+1)!} \). I won’t give all the details - you can try it for yourself. The series rises to fast convergence.

**Solution to (c)**

The proposition holds for \( n = 1 \) since the LHS = \( \frac{x^4(1-x)^4}{1+x^2} \) and the RHS = \( (x^6 \cdot 4x^3 + 5x^4 \cdot 4x^2 + 4) \sum_{k=0}^{n} (-4)^{-k} x^4k(1-x)^4k + \frac{-4}{1+x^2} \)

So we assume that the formula holds for any \( n \).

Then \( \frac{x^4(1-x)^4}{1+x^2} = x^4 \sum_{k=0}^{n} (-4)^{-k} x^4k(1-x)^4k + \frac{-4}{1+x^2} \)

For \( P(x) \sum_{k=0}^{n} (-4)^{-k} x^4k(1-x)^4k \) + \( (-4)^0 P(x) - \frac{4}{1+x^2} \) using the fact that \( \frac{x^4(1-x)^4}{1+x^2} = P(x) - \frac{4}{1+x^2} \) from (b)

Now let \( j = k + 1 \)

Then we have that:

\[
P(x) \sum_{k=0}^{n} (-4)^{-k} x^4k(1-x)^4k + \frac{(-4)^0}{1+x^2} = P(x) \sum_{k=0}^{n} (-4)^{-k} x^4k(1-x)^4k + \frac{(-4)^0}{1+x^2}
\]

Thus the formula is true for \( n + 1 \) and hence true for all \( n \) by induction.

This relationship : \( \pi \cdot \sum_{n=0}^{\infty} \frac{(-1)^n 2n(4n+1)(4n+3)(4n+5)}{8n+1} \sum_{k=0}^{n} \frac{(-4)^{-k} x^4k(1-x)^4k + \frac{-4}{1+x^2}}{1+x^2} \) (820k^3 + 1533k^2 + 902k + 165) provides quick convergence as the following demonstrates. The series approximation is correct to 5 decimal places after using only 2 terms of the series.
11. An HSC calculus example

For each integer \( n \geq 0 \) let \( I_n(x) = \int_0^x e^{-t} \, dt \)

(i) Prove by induction that \( I_n(x) = n! \left( 1 - e^{-x} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \right) \right) \)

(ii) Show that \( 0 \leq \int_0^x e^{-t} \, dt \leq \frac{1}{n+1} \)

(iii) Hence show that \( 0 \leq 1 - e^{-t} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \right) \leq \frac{1}{(n+1)!} \)

(iv) Hence find the limiting value of \( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \) as \( n \to \infty \)

[This was a 4 unit HSC problem. It is actually a very mechanical application of induction]

**Solution to (i)**

The RHS of the formula for \( I_n(x) \) gives \( 1 - e^{-x} \) since \( 0! = 1 \) by definition. So the formula is true for \( n=0 \). Our induction hypothesis is

\[
I_n(x) = n! \left( 1 - e^{-x} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \right) \right)
\]

\[
I_{n+1}(x) = \int_0^x e^{-t} \, dt = \int_0^x e^{-t} \, dt \] \[ \begin{align*}
\text{Remember that } & \int u \, dv = uv - \int v \, du \\
& = -e^{-x} + \int_0^x e^{-t} \, dt \\
& = -x e^{-x} + (n+1) n! \left( 1 - e^{-x} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \right) \right) \text{ using the induction hypothesis} \\
& = -x e^{-x} + (n+1) n! \left( 1 - e^{-x} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \right) \right) \\
& = (n+1)! \left( 1 - e^{-x} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} \right) \right) \\
& = I_{n+1}(x)
\end{align*}
\]

So the formula is true for all \( n \) since it is for \( n=0 \) and the principle of induction shows that if it is true for any \( n \), it is true for \( n+1 \).

**Solution to (ii)**

\[
0 \leq \int_0^x e^{-t} \, dt \leq \int_0^1 e^{-t} \, dt \quad \text{since } e^{-t} \leq 1 \text{ for all } t \in [0, 1]. \text{ Therefore } 0 \leq \int_0^x e^{-t} \, dt \leq \int_0^1 e^{-t} \, dt = \frac{e^{-1}}{n+1} \left| _0^1 \right|_0 = \frac{1}{n+1}
\]

**Solution to (iii)**

\[
0 \leq \int_0^x e^{-t} \, dt \leq \frac{1}{n+1} \quad \text{implies that } \quad 0 \leq n! \left( 1 - e^{-1} \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \right) \right) \leq \frac{1}{n+1} \quad \text{using (i) with } x=1 \text{ substituted}
\]

Hence

\[
0 \leq 1 - e^{-1} \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \right) \leq \frac{1}{(n+1)!} \quad \text{on dividing through by } n!
\]

**Solution to (iv)**

The limit is clearly \( e \) (they wouldn’t have set the problem if it had been something more obscure!) Take any \( \varepsilon > 0 \)

Have to show that \( \exists N \text{ such that } |e - \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \right)| < \varepsilon \text{ for all } n > N. \)
\[ e - (1 + \frac{1}{1^i} + \frac{1}{2^i} + \frac{1}{3^i} + \ldots + \frac{1}{n^i}) = e \left( 1 - e^{-1} \left( 1 + \frac{1}{1^i} + \frac{1}{2^i} + \frac{1}{3^i} + \ldots + \frac{1}{n^i} \right) \right) \leq e \frac{1}{(n+1)!} \]

which can be made arbitrarily small by making n large enough since (n+1)! becomes huge even when n is relatively small. Most mathematicians would leave it here in practice but if you really want to show that you know what you are doing you can go further and actually demonstrate the N.

Indeed, \( \frac{e}{(n+1)!} < \frac{3}{2^n} \) for all \( n \geq 2 \) (why? You could even try a quick inductive proof if you are in the zone!)

So we need \( n \) such that \( \frac{3}{2^n} < \epsilon \) and if \( N = \left\lfloor \frac{\ln(\frac{3}{\epsilon})}{\ln 2} \right\rfloor \) then for all \( n > N \) we will be sure that \( |e - (1 + \frac{1}{1^i} + \frac{1}{2^i} + \frac{1}{3^i} + \ldots + \frac{1}{n^i})| < \epsilon \).

If you don’t believe it just try it with \( \epsilon = 10^{-6} \) say. \( N \) will be 22 and \( \frac{e}{(n+1)!} = \frac{e}{23!} \) which is clearly \( \ll 10^{-6} \).

### 12. A diagonal matrix example

In the theory of similar matrices one obtains expressions such as: \( A^n = \Omega^n S^{-1} \) where \( \Omega \) is a diagonal matrix of eigenvalues \( \lambda_i \). In other words \( \Omega \) is an \( m \times m \) matrix whose elements \((\Omega_{ij})\) are:

\[
\Omega_{ij} = \begin{cases} 
\lambda_i & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases}
\]

\[
\Omega = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_{m-1} & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda_m
\end{pmatrix}
\]

Prove by induction for any diagonal matrix \( \Omega \), \( \Omega^n = \sum_{k=1}^{n} \lambda_i^n \) for all integers \( n \geq 1 \).

**Solution**

It quickly becomes obvious that this proposition is true when you start multiplying matrices. However, let’s retrace how to multiply matrices.

The \((i,j)\)th element of the product of two matrices \( A \) and \( B = \sum_{k=1}^{n} a_{ik} b_{kj} \). Look very carefully at this formula - if you don’t understand it, stop here and work out why it works. It will pay dividends later many times over.

The \((i,j)\)th element of the product is the sum of the \( j \)th row of \( A \) and multiplying by the \( i \)th column of \( B \). Rows go into columns - if you remember that one you will be able to multiply matrices.

Clearly the proposition is true for \( n=1 \). The more interesting basis step is when \( n=2 \) because that gives a hint about how the proof is generalised. What is the \((i,j)\)th element of \( \Omega^2 \)?

\((i,j)\)th element of \( \Omega^2 = \Omega \Omega \) (note that because \( k \) is repeated it is ‘bound’). But if \( i > j \) then \( \Omega_{ik} \Omega_{kj} = 0 \) (why? Because \( \Omega_{nm} = 0 \) for \( l > m \)). Similarly if \( i < j \) then \( \Omega_{ik} \Omega_{kj} = 0 \) (why? Because \( \Omega_{nn} = 0 \) for \( i < n \)). So the only contributors to the sum are given by \( i=j \) \( \lambda_i^2 \). This holds for \( 1 \leq i \leq n \). What this means is that \( \Omega^2 \) has zero elements off the diagonal and on the diagonal the elements are \( \lambda_i^2 \). So the proposition is true for \( n=2 \).

Suppose the proposition is true for \( n \). Now \( \Omega^{n+1} = \Omega^n \Omega \)

So the \((i,j)\)th element of \( \Omega^{n+1} = \sum_{k=1}^{n} \lambda_i^k \lambda_j^k \) using the induction hypothesis (ie the non-zero \((i,k)\)th element of \( \Omega^n \) is \( \lambda_i^k \)).

Therefore \((i,j)\)th element of \( \Omega^{n+1} = \sum_{k=1}^{n} \lambda_i^k \lambda_j^k = \lambda_i^k \lambda_j^k = \lambda_{i+j}^{k+1} \) using the fact that \( \Omega_{ij} = 0 \) when \( k > j \). This is what we wanted to prove.

### 13. An example from actuarial practice

Actuarial mathematics contains a number of basic building blocks which are used to value cashflows occurring at various points in time. Consider \$1 which is invested at the end of each year for \( n \) years at an effective rate of interest of \( i \) per annum. This means that \$1 accumulates to \$\left(1+i \right)^n \) after \( n \) years. What is the accumulation of this stream of \( n \) investments of \$1 after \( n \) years? The actuaries have a
Prove this formula for all integers n 1

Now for a generalisation of the concepts consider n investments of $1 made at times t =0, 1, 2, ..., n such that all investment income is reinvested at the end of each period. The interest rate for period [j-1, j] is $i_{j-1}$ for $j=1, 2, ..., n$. Derive a formula for the accumulation of these investments to the end of year n and then prove it by induction. Next show that if the interest rate is constant at i for all periods, you get $S(n, i) = \frac{(1+i)^{n+1}}{1-i}$.

Solution

What is the accumulation of this stream of n investments of $1 after n years? Let $S(n, i)$ be the amount accumulated to the end of year n at effective rate i per annum.

The first payment of $1 grows to $(1+i)S(n, i)$

... The kth payment of $1 grows to $(1+i)^{k-1}S(n, i)$

The nth payment of $1 grows to $1 since it occurs at the end of period n.

So $S(n, i) = \sum_{j=1}^{n+1} (1+i)^{j-1}$ (this is simply the sum of the above amounts). This is just a geometric series so we use the old trick that Cauchy actually used in his calculations on series:

$(1+i) S(n, i) = (1+i) \sum_{j=1}^{n} (1+i)^{j-1} = \sum_{j=1}^{n+1} (1+i)^{j-1}$

Let $v = \frac{1}{1+i}$

Then $S(n, i) = \sum_{j=1}^{n} (1+i)^{j-1} = (1+i)^{n} \sum_{j=1}^{n} v^{j}$

Now Cauchy's old trick was this: let $X = \sum_{j=1}^{n} v^{j}$ then $X - vX = vX = v^{n+1}$ (if you can't see this write X and vX out)

Hence $X = \frac{v-v^{n+1}}{1-v} = \frac{v}{1-v}(1-v^{n}) = \frac{\frac{1}{1+i}-1}{i(1+i)}$ since $\frac{1}{1-v} = \frac{1}{i}$

Thus $S(n, i) = (1+i) \sum_{j=1}^{n} v^{j} = (1+i)^{n} \left[ \frac{\frac{1}{1+i}-1}{i} \right] = \frac{(1+i)^{n+1}}{1-i}$

Is the formula for $S(n, i)$ true for all positive integers n? It is clearly true for n = 1 since $1$ invested at the end of year 1 is just $1$ - the formula gives:

$S(1, i) = \frac{(1+i)^{1}}{1-i} = 1$

Assume $S(n, i) = \frac{(1+i)^{n}}{1-i}$ for any integer n 1

The basic economics of the problem ensure that:

$S(n+1, i) = S(n, i)$ accumulated for 1 year + $1$ at the end of year n+1

= $(1+i) S(n, i) + 1$

= $(1+i) \left( \frac{(1+i)^{n}}{1-i} \right) + 1$ (using the induction hypothesis)

= $\frac{(1+i)^{n+1} - 1}{1-i} + 1$

= $\frac{(1+i)^{n+1} - (1-i)}{1-i}$

= $\frac{(1+i)^{n+1} - 1}{1-i}$

So the formula is true for n+1 and hence the formula is true by induction.

Extension: What is the formula if the payments are made at the beginning of the year?

The following diagram enables you to see what the formula is without any algebra:

---

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The top diagram show the accumulation where payments are at the end of each year. The bottom diagram shows the translation of the top set of payments to the left by 1 year. Thus that accumulation, which we know to be $S(n,i)$ has 1 more year to accumulate and the amount must be $(1+i)S(n,i)$. If you doubt this then do the algebra and you will get the following:

$$\tilde{S}(n, \bar{i}) = (1+\bar{i})^n + (1+\bar{i})^{n-1} + \ldots + (1+\bar{i}) = (1+\bar{i}) \left[ (1+\bar{i})^{n-1} + \ldots + 1 \right] = (1+\bar{i}) S(n, i)$$

Generalisation to non-constant interest rates per period.

Now for a generalisation of the concepts consider $n$ investments of $1$ made at times $t = 0, 1, 2, \ldots, n-1$ such that all investment income is reinvested at the end of each period. The interest rate for period $[j-1, j]$ is $i_{j-1}$ for $j=1,2,\ldots,n$. Derive a formula for the accumulation of these investments to the end of year $n$ and then prove it by induction. Next show that if the interest rate is constant at $i$ for all periods, you get $S(n, \bar{i}) = \frac{(1+\bar{i})^{n-1} - 1}{\bar{i}}$.

To get a feel for the structure of the formula take a low order case of 3 periods.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1+i_0$</td>
<td>$1+i_0+(1+i_0)i_1$</td>
<td>$(1+i_0)(1+i_1)$</td>
<td>$1+i_1+i_2$</td>
</tr>
<tr>
<td>1</td>
<td>$1+i_1$</td>
<td>$1+i_1+(1+i_1)i_2$</td>
<td>$(1+i_1)(1+i_2)$</td>
<td>$1+i_2$</td>
</tr>
</tbody>
</table>

Total: $1 + i_2 + (1+i_1)(1+i_2) + (1+i_0)(1+i_1)(1+i_2)$

This suggests that our formula should look like this: Accumulation to the end of year $n = A_n = 1 + i_{n-1} + (1+i_{n-1})(1+i_{n-2}) + (1+i_{n-1})(1+i_{n-2}) + (1+i_{n-3}) + \ldots + (1+i_1)(1+i_0)(1+i_0)$

Our formula is trivially true for $n=1$ since $A_1 = 1 + i_0$. A more interesting test of the accuracy of the base case is for $n=2$. The formula gives $A_2 = 1 + i_1 + (1+i_1)(1+i_0)$. Going back to basics $1$ invested at $t=0$ gives rise to $1+i_0$ at $t=1$ and the interest on this at $t=2$ is $(1+i_0)i_1$ so the $1$ accumulates to $1+i_0+(1+i_0)i_1 = (1+i_0)(1+i_1)$ at $t=2$. The remaining $1$ is invested at $t=1$ and it accumulates to $1+i_1$ at $t=2$. Thus the total accumulation at $t=2$ is $(1+i_0)(1+i_1) + 1+i_1 = A_2$ as claimed. So we have proved the
formula for the base cases of \( n = 1 \) and \( 2 \). Let’s assume the formula is true for any \( n \) ie \( \text{A}_n = 1 + i_{n-1} + (1 + i_{n-2})(1 + i_{n-2}) + (1 + i_{n-3})(1 + i_{n-2}) + ... + (1 + i_{n-k})(1 + i_{n-k}) \) at time \( n \). The last payment at time \( n+1 \). We know that the PV of the first \( n \) payments is \( \text{A}(n,i) \) so:

\[
\text{A}(n+1, i) = \text{present value of stream of } n+1 \text{ $1 amounts payable and the end of each year evaluated at effective rate } i \text{ per annum.}
\]

Suppose \( \text{A}(n,i) \) is true for any \( n \).

If the interest rate in every period is constant ie \( i = \text{constant} \), then \( \text{A}(n,i) \) is true for \( n+1 \) and hence true for all \( n \) since it is true for \( n = 1 \).

Thus the formula is true for \( n+1 \) and hence true for all \( n \) since it is true for \( n = 1 \).

If the interest rate in every period is constant ie \( i = \text{constant} \), then \( \text{A}(n,i) \) is true for \( n+1 \) and hence true for all \( n \) since it is true for \( n = 1 \).

14. A further example from actuarial practice

In the world of finance whatever goes up can come down and actuaries also have to work out the present value (PV) of income streams.

Consider \( \frac{1}{1+i} \) invested today at effective rate \( i \) per annum for 1 year. At the end of the year this initial amount has grown to \( \frac{1}{1+i} + \frac{1}{1+i} = \frac{1}{1+i} \) s. Thus the present value (ie today’s value) of \( S \) due in one year’s time at effective rate \( i \) per annum is \( \frac{1}{1+i} = \text{(1 + i)}^{-1} \).

Now, consider a stream of \( n \) $1 payments, the first of which is made at the end of year 1, the second at the end of year 2 and so on for \( n \) years. The actuaries have a symbol for the present value of these \( n \) payments: \( \text{A}(n,i) = \frac{1-v^n}{i} = \frac{1-(1+i)^{-n}}{i} \) where \( v = \frac{1}{1+i} \).

Prove the annuity formula for all integers \( n \)

Having done this problem and the earlier one you should be able to derive a relationship between \( S(n,i) \) and \( \text{A}(n,i) \). In essence, having done these two problems you will know the basics of most valuation techniques.

If the payments of \( S \) are made at the beginning of the year rather than the end, show that the annuity formula for the advanced set of payments is:

\[
\text{A}(n,i) = (1 + i) \text{A}(n, i)
\]

Solution

\[
\text{A}(n,i) = (1 + i)^{-1} + (1 + i)^{-2} + (1 + i)^{-3} + ... + (1 + i)^{-(n-1)} + (1 + i)^{-n} = \sum_{k=1}^{n} (1 + i)^{-k}
\]

Once again this is a standard geometric series which can be valued this way:

\[
(1+i) \text{A}(n,i) = 1 + (1 + i)^{-1} + (1 + i)^{3} + (1 + i)^{-3} + ... + (1 + i)^{-(n-1)} = \sum_{k=1}^{n} (1 + i)^{-k}
\]

Therefore \( (1+i) \text{A}(n,i) - \text{A}(n,i) = 1 - v^n \)

Hence, \( \text{A}(n,i) = 1 - v^n \)

Finally, \( \text{A}(n) = \frac{1-v^n}{i} \)

We need to prove this inductively for integers \( n \geq 1 \).

\[
\text{A}(1,i) = \frac{1-v}{i} = \frac{1 - \frac{1}{1+i}}{i} = \frac{1}{1+i} \frac{1}{1+i} = v
\]

But \( \text{A}(1,i) \) is the present value of \( S \) due in 1 year’s time and if we invest \( \frac{1}{1+i} \) today at effective interest rate \( i \) the accumulated amount in 1 year will be \( \frac{1}{1+i} + \frac{1}{1+i} \) \( i = \frac{1}{1+i} + \frac{1}{1+i} = 1 \).

This means that \( v = \frac{1}{1+i} \) is the present value of 1 ie \( v = \text{A}(1,i) \).

Suppose \( \text{A}(n,i) \) is true for any \( n \).

\( \text{A}(n+1,i) \) = present value of stream of \( n+1 \) $1 amounts payable and the end of each year evaluated at effective rate \( i \) per annum. Focus on the last payment at time \( n+1 \). We know that the PV of the first \( n \) payments is \( \text{A}(n,i) \) so:
\[ A(n+1,i) = A(n,i) + v^{n+1} \]

\[ = \frac{1 - v}{i} + v^{n+1} \] using the induction hypothesis

\[ = \frac{1 - v(1 - i)}{i} \]

\[ = \frac{1 - v}{1 - i} \]

\[ = \frac{1}{1 - i} \]

So the formula is true for \( n+1 \) and hence is true by induction. What is the relationship between \( S(n,i) \) and \( A(n,i) \)? It should be clear that since \( A(n,i) \) is the present value of the stream of payments for \( n \) years its accumulation to the end of year \( n \) must be \((1 + i)^n \ A(n, i). \) If you doubt this reasoning just note that \((1 + i)^n \ A(n, i) = \frac{(1 + i)^n \ [1 - (1 + i)^{-n}]}{i} = \frac{(1 + i)^n - 1}{i} = S(n,i)\)

**Payments at the beginning rather than the end.**

As the following diagram demonstrates the two streams are simply translated by one time period:

The present value of \( n \) payments made at the beginning of each of \( n \) periods is:

\[ A(n,i) = \frac{(1 - v^n)}{i} \]

\[ \begin{array}{cccccccc}
0 & 1 & 2 & 3 & \cdots & \cdots & \cdots & N - 1 & N \\
\$1 & \$1 & \$1 & \cdots & \cdots & \cdots & \$1 & \$1
\end{array} \]

\[ \hat{A} (n,i) = (1+i) \ A(n,i) \]

\[ \begin{array}{cccccccc}
0 & 1 & 2 & \cdots & \cdots & \cdots & N - 1 & N \\
\$1 & \$1 & \$1 & \cdots & \cdots & \cdots & \$1
\end{array} \]

The present value of \( n \) payments made at the beginning of each of \( n \) periods is:

\[ A(n,i) = 1 - v^n + A(n, i) \] ie

you add the present value of the first payment at time 0 and subtract the present value of the payment at time \( n \) which is \( v^n \) where \( v = \frac{1}{1+i} \)

But \( A(n,i) = \frac{1 - v^n}{i} \) so we have:

\[ \hat{A}(n, i) = i \ A(n, i) + A(n, i) = (1 + i) \ A(n, i) \]

You could of course sum the series from first principles and you will get the same answer. It goes like this:

\[ \hat{A}(n, i) = 1 + v + v^2 + \ldots + v^{n-1} \]

\[ v \ A(n, i) = v + v^2 + \ldots + v^{n-1} + v^n \]
(1 - \nu) A(n, i) = 1 - \nu^n

So \ A(n, i) = \frac{1 - \nu^n}{1 - \nu} = \frac{1 - \nu^n}{1 - \frac{1}{1 + i}} = (1 + i) \frac{1 - \nu^n}{i} = (1 + i) A(n, i) as promised.

Another way of looking at \ A(n, i) is that it equals the present value of a payment of $1 at time 0 plus the present value of (n - 1) payments of $1 payable at the end of years 2, 3, ..., n (look at the diagram and it all becomes clear). Thus \ A(n, i) = 1 + A(n - 1, i). Don’t believe it?

1 + A(n - 1, i) = 1 + \frac{1 - \nu^{n-1}}{i} = \frac{1 + i - \nu^{n-1}}{i} = \frac{\nu^{n-1} - \nu}{i} = (1 + i) A(n, i) = A(n, i)

You should be convinced now that there are several ways to approach such valuation problems.

15. A more complicated combinatorial problem

Let \ n be a positive integer. Prove by induction that:

S(n) = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{k} C^n_k = 1 + \frac{1}{2} + \ldots + \frac{1}{n}

You will need to observe that \frac{1}{n+1} C^n_{n+1} = \frac{1}{k} C^n_{k-1} so do a one line proof of that along the way.

Solution

Doing this “cold” might lose a lot of students but if you have followed the combinatorial problems given above there is only one missing piece and it involves proving that \frac{1}{n+1} C^n_{n+1} = \frac{1}{k} C^n_{k-1}

Let’s get that out of the way first. \ LHS = \frac{(n+1)!}{(n+1)! (n+1)!} = \frac{n!}{k! (n-k)! (k-1)!} = \frac{1}{k} C^n_{k-1}

S(1) is true since 1 = \sum_{k=1}^{1} \frac{(-1)^{n-k}}{k} C^n_k

Assume S(n) is true for any \ n and consider \ S(n+1) = \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} C^n_k = \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} (C^n_k + C^n_{k+1})

by Pascal’s Identity

= \sum_{k=1}^{n} \frac{(-1)^{n-k}}{k} C^n_k + \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} C^n_{k+1}

because C^n_{n+1} = 0

= S(n) + \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} C^n_{k+1}

using the induction hypothesis and our one line result proved at the outset. Now we need to show that the last term = \frac{1}{n+1} and we are done.

Now it was proved earlier (Problem 8) that \sum_{k=0}^{n} (-1)^k C^n_k = 0 \ for any \ n \ and so \ \sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} C^n_{k+1} = \frac{-1}{n+1} \sum_{k=1}^{n+1} (-1)^k C^n_{k+1} = \frac{-1}{n+1} \left( \sum_{k=0}^{n} (-1)^k C^n_k - 1 \right) = \frac{-1}{n+1} - (0 - 1) = \frac{1}{n+1}.

So S(n+1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n+1} as required.

If you have trouble following the last bit of algebra write it out like this:

\sum_{k=1}^{n+1} \frac{(-1)^{n-k}}{k} C^n_k = \frac{1}{n+1} \left( C^n_{n+1} - C^n_0 + C^n_1 + \ldots + (-1)^n \right) = \frac{1}{n+1} \left( - \sum_{k=0}^{n} (-1)^k C^n_{k+1} + 1 \right) = \frac{1}{n+1} (0 + 1) = \frac{1}{n+1}

16. Working out your retirement savings

The Australian superannuation industry is a trillion dollar business and is large even by international standards (we rate number four). The growth of the Australian system has been driven largely by compulsory superannuation and, of course, remarkable stock market growth over many years in recent times. A very common question people have is “How much do I have to save to generate a retirement income of such and such per year?”. Superannuation funds produce calculators to enable people to perform that type of calculation.

To set the problem up the building blocks are as follows.

(1) You assume the person has \ n years to save.

(2) You assume that an initial tax deductible contribution of \ C \ is made (either by an employer or a self employed person) and this escalates each year by \ \alpha \%. Under the law employers have to contribute 9\% of salary so that \ C \approx 9\% of salary in the first year. In effect you are assuming that salary is increasing on a compound basis of \alpha\% pa. It is possible to make contributions out of after-tax money and
these are not taxed within the fund but we will ignore them in this simple model.

(3) Tax of 15% is assumed to be deducted from each payment covered by (2)

(4) The contributions are assumed to be made in the middle of the year and the tax is assumed to be deducted at the same time. One can assume any other periodicity but the “accuracy” of such assumptions is illusory because the model assumes fixed earnings rates and has no probabilistic dimension to it. Markets have significant variability over the time scales involved in superannuation eg up to 40 years or more in theory. In other words it is completely unrealistic yet this is precisely the sort of model that is commercially available (indeed I have built on-line calculators for a major superannuation fund which are not much more complicated than the one involved in this problem).

(5) You assume a fixed earnings rate of i % pa in the invested fund of contributions. As noted above this is unrealistic but it is a common assumption because the public does not understand probabilistic (stochastic) investment models. This rate is assumed to be an after-tax rate. When superannuation investments are accumulating (ie before the pension is paid) the earnings attract a maximum rate of tax of 15%. When the pension begins to be paid there is no tax on the fund investment earnings. The actual determination of the after-tax rate of return for various types of investment portfolios (which can include shares, property, bonds, infrastructure etc) is a statistical question which we will avoid. We just assume the compound after tax earnings rate is i% pa.

The problem

(i) Show that with the above assumptions the accumulated amount at the end of the nth year is A(n) where:

\[ A(n) = \sum_{k=1}^{n} C(1 + a)^{n-k} \]  

Note that you can take \( C = 0.85 \) of the initial contribution (C) since it is a constant for the problem

(ii) Prove this formula by induction.

(iii) Build an Excel spreadsheet which will verify the formula from first principles for an initial contribution of $1,000, a = 4\% , i = 6\% and n = 10 years. In other words you have to show how the fund would accumulate on an annual basis. You should get $13,590.90 by both the formula and the first principles approach.

(iv) Now that you know how to accumulate you need to be able to work out what annual pension amount could be paid to the pensioner. All of this will be in future dollars since we have not allowed for inflation. If the accumulated amount at the end of year \( n \) is \( A(n) \) and the pensioner’s life expectancy at that time is \( L \) derive a formula for the annual pension, payable at the beginning of the year.You can assume the rate of investment return in pension phase is \( \frac{0.85}{1+0.05} \) which crudely accounts for the fact that there is no tax in pension phase. Discounting the figures by an assumed constant rate of inflation (\( j \% \) pa) will give you the value of the pension in today’s dollars.

Numerical example: Assume \( C = 2,000 \), \( a = 4\% \), \( i = 6\% \), \( n = 40 \), \( L = 25 \) and \( j = 3\% \). Work out the annual pension amount in today’s dollars.

Solution

The \( k^{th} \) contribution is \( C_k = C(1 + a)^{k-1} \) eg \( C_1 = C \).

\[ A(n) = C(1 + a) - \frac{1}{i} [ (1 + a)^{n-1} - 1 ] + C(1 + a)^2 - \frac{1}{i} [(1 + a)^{n-2} - 1] + ... + C(1 + a)^{n-1} - \frac{1}{i} (1 + a)^1 \]  

Note that in this step we are using the convention that \( C = 0.85 \times \) initial contribution.

The following diagram shows visually what is going on:

\[ C \quad C(1+a) \quad C(1+a)^{n-1} \]

\[ 0 \quad 1 \quad 2 \quad ................. \quad n-1 \quad n \]

We can write \( A(n) \) in sigma notation as follows:

\[ A(n) = \sum_{j=1}^{n} C(1 + a)^{n-j-1} (1 + i)^{-j} = \sum_{j=1}^{n} C(1 + a)^{j-1} (1 + i)^{-j} \]

\[ = C(1 + a)^{-1} (1 + i)^{-1} \sum_{j=1}^{n} \left( \frac{1+a}{1+i} \right)^j \]  

where we have moved the factors independent of \( j \) out of the sigma function.

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We are nearly there since all we have to do is sum \( \sum_{j=1}^n u^j \) where \( u = \frac{1 + a}{1 + i} \).

The sum is simply \( \frac{u(1 - u^n)}{1 - u} \). If you can’t see why the let \( S = u + u^2 + \ldots + u^{n-1} + u^n \) then \( uS = u^2 + \ldots + u^n + u^{n+1} \)

\[ S(1 - u) = u - u^{n+1} \quad \text{so} \quad S = \frac{u(1 - u^n)}{1 - u}. \]

Finally, \( A(n) = C(1 + a)^{-1}(1 + \delta^{n + \frac{1}{2}} \frac{u(1 - u^n)}{1 - u}). \)

\[ = C(1 + \delta^{n + \frac{1}{2}} \frac{1 - u^n}{1 - u}) \quad \text{as required.} \]

(ii) Prove this formula by induction.

\( A(1) = C(1 + \delta^{\frac{1}{2}} \) since the initial after-tax contribution of \( C \) is accumulated for half a year.

The formula gives \( A(1) = C(1 + \delta^{\frac{1}{2}} \) so the formula is true for \( n = 1 \).

Now \( A(n + 1) = A(n) \) accumulated for 1 year + \( C(1 + a)^n \) accumulated for half a year

\[ = (1 + i)A(n) + C(1 + a)^n(1 + \delta^{\frac{1}{2}} \]

\[ = (1 + i)C(1 + \delta^{n + \frac{1}{2}} \frac{1 - u^n}{1 - u}) + C(1 + a)^n(1 + \delta^{\frac{1}{2}} \]

\[ = C(1 + \delta^{n + \frac{1}{2}} \frac{1 - u^n}{1 - u}) + C(1 + a)^n(1 + \delta^{\frac{1}{2}} \]

\[ = C(1 + \delta^{n + \frac{1}{2}} \frac{1 - u^n}{1 - u}) + C(1 + \delta^{n + \frac{1}{2}} \frac{1 - u^n}{1 - u}) \]

where we have used the fact that \( u = \frac{1 + a}{1 + i} \) and so \( (1 + \delta^n u^n) = (1 + a)^n \)

\[ = C(1 + \delta^{n + \frac{1}{2}} \frac{1 - u^n}{1 - u}) \quad \text{establishing that the formula is true for} \ n + 1. \]

(iii) Build an Excel spreadsheet which will verify the formula from first principles for an initial contribution of \$1,000 , \ a = 4\% , \ i = 6\% \) and \( n = 10 \) years. In other words you have to show how the fund would accumulate on an annual basis. You should get \$13,590.90 by both the formula and the first principles approach.
Note that Cell C7 is 0.85 x initial contribution

(iv) Now that you know how to accumulate you need to be able to work out what annual pension amount could be paid to the pensioner. All of this will be in future dollars since we have not allowed for inflation. If the accumulated amount at the end of year n is A(n) and the pensioner’s life expectancy at that time is L derive a formula for the annual pension, payable at the beginning of the year. You can assume the rate of investment return in pension phase is $i %$ which crudely accounts for the fact that there is no tax in pension phase ($i %$ is the after tax rate of return in accumulation phase). Discounting the figures by an assumed constant rate of inflation ($j %$ pa) will give you the value of the pension in today’s dollars. Numerical example: Assume $C = 2,000, a = 4%, i = 6%, n = 40, L = 25$ and $j = 3%$. Work out the annual pension amount in today’s dollars.

From (i) we know that $A(n) = \frac{C}{L} \left( 1 - \frac{(1+u)^n}{1-u} \right)$. This amount must support a pension of $P$ payable annually in advance for L years at rate $r = \frac{i}{0.85}$. The formula for an annuity of $1$ payable annually in arrears for L years at effective rate $r %$ pa is $\frac{1 - (1 + r)^{-L}}{r}$ but we want the formula for annual payments made at the beginning of the year. From an earlier problem the value of such an annuity is $(1 + r) \left[ \frac{1 - (1 + r)^{-L}}{r} \right]$. Thus at time $n$ (when this valuation is taking place) $A(n) = P(1 + r) \left[ \frac{1 - (1 + r)^{-L}}{r} \right]$ so $P$ can be solved. The pension in today’s dollars is therefore $1.03^{-40} P$.

When you plug the numbers in you will get an annual pension payment in today’s dollars of $11,856.36 which is LESS than the current Federal government age pension for a single person. Moreover it is not indexed for inflation so over the expected term of the pension the pension has even less purchasing power. This is very little to live on and explains the great emphasis on saving more. This level of income is well below the poverty line. The following shows you how the pension is drawn down to zero over 25 years (the value of the annual pension payment at year 40 is $38,676).
17. Leibniz’s rule

Leibniz’s rule is as follows:

\[ \frac{d^n(uv)}{dx^n} = \sum_{i=0}^{n} C^n_i \frac{d^i u}{dx^i} \frac{d^{n-i} v}{dx^{n-i}} \]

You have to assume that \( \frac{d^0 u}{dx^0} = u \) and \( \frac{d^0 v}{dx^0} = v \)

Prove Leibniz’s rule by induction for all integers \( n \) 1. Along the way you will need to use the result proved above, i.e. \( C_{i+1} = C_i + C_{i-1} \) (Pascal’s Identity)

Solution

The formula is true for \( n = 1 \) since

\[ \frac{d^1(uv)}{dx^1} = u \frac{d v}{dx} + v \frac{d u}{dx} \]  \( (1) \)

and

\[ \sum_{i=0}^{1} C^1_i \frac{d^i u}{dx^i} \frac{d^{1-i} v}{dx^{1-i}} = 1 \frac{d u}{dx} \frac{d v}{dx} + \frac{d u}{dx} \frac{d v}{dx} \]

\[ = u \frac{d v}{dx} + v \frac{d u}{dx} \]

Assume the formula is true for all \( n \). Then:

\[ \frac{d^{n+1}(uv)}{dx^{n+1}} = \frac{d}{dx} \left( \frac{d^n(uv)}{dx^n} \right) \]

\[ = \frac{d}{dx} \left( \sum_{i=0}^{n} C^n_i \frac{d^i u}{dx^i} \frac{d^{n-i} v}{dx^{n-i}} \right) \]

\[ = \sum_{i=0}^{n} C^n_i \frac{d^{i+1} u}{dx^{i+1}} \frac{d^{n-i} v}{dx^{n-i}} + \sum_{i=0}^{n} C^n_i \frac{d^i u}{dx^i} \frac{d^{n-i+1} v}{dx^{n-i+1}} \]

\[ = A + B \]  \( (2) \)

Because formula (1) is nothing more than a variant of the binomial theorem we should be able to use the same trick as we used before in the proof of the binomial theorem. Namely we are ultimately looking for this:

\[ \sum_{i=0}^{n} C^n_i \frac{d^i u}{dx^i} \frac{d^{n-i} v}{dx^{n-i}} \]  \( and we will use the fact that \)

\[ C_{i+1} = C_i + C_{i-1} \]

Going back to (2):

\[ A = \sum_{i=0}^{n} C^n_i \frac{d^{i+1} u}{dx^{i+1}} \frac{d^{n-i} v}{dx^{n-i}} \]

Let \( j = i + 1 \)

\[ A = \sum_{j=1}^{n+1} C^n_{j-1} \frac{d^{j} u}{dx^{j}} \frac{d^{n-j+1} v}{dx^{n-j+1}} \]

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\[
\sum_{j=0}^{n-1} C^0_j \frac{d^{n-1-j} u}{dx^{n-1-j}} \frac{dv}{dx}
\]

\[
\sum_{j=0}^{n-1} C^0_j \frac{d^{n-1-j} u}{dx^{n-1-j}} \frac{dv}{dx} \quad \text{since } C^n_j = 0 \text{ for } j < 0
\]

Now \( B = \sum_{j=0}^{n} C^n_j \frac{d^{n-j} u}{dx^{n-j}} \frac{dv}{dx} \) and just change the index variable to \( j \) and you get

\[
B = \sum_{j=0}^{n} C^n_j \frac{d^{n-j} u}{dx^{n-j}} \frac{dv}{dx}
\]

\[
= \sum_{j=0}^{n} C^n_j \frac{d^{n-j} u}{dx^{n-j}} \frac{dv}{dx} \quad \text{since } C^n_{n+1} = 0
\]

So

\[
\frac{d^{n-1} (u v)}{dx^{n-1}} = A + B = \sum_{j=0}^{n-1} C^0_j \frac{d^{n-1-j} u}{dx^{n-1-j}} \frac{dv}{dx} + \sum_{j=0}^{n-1} C^n_j \frac{d^{n-1-j} u}{dx^{n-1-j}} \frac{dv}{dx}
\]

\[
= \sum_{j=0}^{n-1} \left( C^0_j + C^n_j \right) \frac{d^{n-1-j} u}{dx^{n-1-j}} \frac{dv}{dx}
\]

\[
= \sum_{j=0}^{n-1} C^0_j \frac{d^{n-1-j} u}{dx^{n-1-j}} \frac{dv}{dx}
\]

Hence the formula is true for \( n + 1 \) and so it is true by the principle of induction.

### 18. An example of a recurrence relation

You are told that a sequence has the following properties: \( u_{n+3} = 6 u_{n+2} - 5 u_{n+1} \) where \( u_1 = 2 \) and \( u_2 = 6 \).

By inspired guessing you claim that \( u_n = a b^{n+k} + c \) where \( a > 0 \) and \( b > 0 \). How would you find the values for \( a, b, c \) and \( k \) (which is a fixed integer) and then verify your guess?

**Solution**

\[ u_{n+3} = ab^{n+3+k} + c = 6(ab^{n+2+k} + c) - 5(ab^{n+1+k} + c) \]

\[ ab^{n+3+k} + c = ab^{n+1+k} (6b - 5) + c \]

So \[ ab^{n+3+k} = ab^{n+1+k} (6b - 5) \]

Therefore \( b^2 - 6b + 5 = 0 \) since \( a > 0 \) and \( b > 0 \). Thus \( (b - 5) (b - 1) = 0 \) i.e. \( b = 5 \) or \( b = 1 \).

Try \( b = 1 \). In that case \( u_1 = a + c = 2 \)

\[ u_2 = a + c = 6 \rightarrow \text{a contradiction.} \]

So \[ b = 5 \] and \( u_n = a 5^{n-k} + c \). Since \( u_1 = 2 \), \( a 5^{1-k} + c = 2 \) and \( u_2 = a 5^{2-k} + c = 6 \).

We have two equations:

\[ a 5^{1+k} + c = 2 \quad ...(1) \]

\[ a 5^{2+k} + c = 6 \quad ...(2) \]

Multiply equation (1) by 5:

\[ a 5^{2+k} + 5c = 10 \quad ...(3) \]

Subtract (2) from (3):

\[ 4c = 4 \]

So \( c = 1 \)

So \( u_0 = a 5^{0+k} + 1 \) and hence \( u_1 = a 5^{1+k} + 1 = 2 \Rightarrow a 5^{1+k} = 1 \Rightarrow a = 5^{-1-k} \)

Finally, \( u_0 = a 5^{0+k} + 1 = 5^{-1-k} 5^{0+k} + 1 = 5^{n-1} + 1 \)

So the inspired guess is: \( u_n = 5^{n-1} + 1 \) for \( n \geq 1 \)

But is it correct? Let’s prove it by induction.

\[ u_1 = 5^0 + 1 = 2 \text{ which is correct.} \]

\[ u_2 = 5^1 + 1 = 6 \text{ which is also correct.} \]
Assume \( u_{n+3} = 6u_{n+2} - 5u_{n+1} \) and \( u_n = 5^{n+1} + 1 \) are true for all positive integers \( n \geq 3 \) where \( n \geq 1 \)

Therefore \( u_{n+4} = 6u_{n+3} - 5u_{n+2} \)

\[
= 6(5^{n+2} + 1) - 5(5^{n+1} + 1) \\
= 30 \times 5^{n+1} + 6 - 5 \times 5^{n+1} + 1 \\
= 25 \times 5^{n+1} + 1 \\
= 5^{n+3} + 1
\]

So the inspired guess is true for \( n+4 \) and since it is true for \( n=1 \) and \( n=2 \) and for all positive integers \( n \leq n+3 \) (where \( n \geq 1 \)) it is true for all \( n \).

**19. Another combinatorial example**

Prove that: \( C_r^{r+1} + C_r^{r+2} + ... + C_r^{r+m} = C_{r+1}^{r+m+1} \) where \( r, m \geq 1 \)

Here \( r \) is fixed and \( m \) varies so you will need to base the inductive proof on \( m \).

**Solution**

\[
C_r^{r+1} + C_r^{r+2} + ... + C_r^{r+m} = C_{r+1}^{r+m+1} \quad (1)
\]

When \( m=1 \), LHS of (1) is \( 1 + r + 1 = r + 2 = C_{r+1}^{r+2} \)

Assuming that (1) is true for all \( m \leq k \) we have that:

\[
C_r^{r+1} + ... + C_r^{r+k+1} = [C_r^{r+1} + ... + C_r^{r+k}] + C_r^{r+k+1}
\]

\[
= C_{r+1}^{r+k+1} \text{ using the inductive hypothesis}
\]

\[
= C_{r+1}^{r+k+1} \text{ using Pascal's Identity}
\]

Accordingly (1) is proved true by induction.

**20. When not to use induction**

There are many instances where it is actually easier and more insightful to use a combinatorial style of proof rather than an inductive one. A typical example is this:

\[
\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n} \quad \text{for integral } n \geq 1
\]

To prove this inductively you would have to establish a useful relationship between \( \binom{n+1}{k}^2 \) and \( \binom{n}{k}^2 \) in order to use the inductive hypothesis. Pascal's Identity is a possibility here since \( \binom{n+1}{k} = \binom{n}{k} \binom{n}{k-1} \) but \( \binom{n+1}{k}^2 = \binom{n}{k}^2 + 2\binom{n}{k}\binom{n}{k-1} \)

\( \binom{n}{k-1}^2 \) you would have to find a further relationship for the sum of the products of \( \binom{n}{k} \binom{n}{k-1} \). In effect this amounts to proving a sub-case of an identity relating to the hypogeometric series ie:

\[
\binom{n}{0} \binom{m}{k} + \binom{n}{1} \binom{m}{k-1} + ... + \binom{n}{k-1} \binom{m}{1} + \binom{n}{k} \binom{m}{0} = \binom{n+m}{k} \]

This can be done by an inductive approach but not very insightfully.

So your challenge is to prove:

\[
\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + ... + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n} \quad \text{(for integral } n \geq 1) \text{ by a combinatorial approach and then by appeal to the binomial theorem.}
\]

**Solution**
First a combinatorial approach.

The RHS of \( \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n} \) is simply the number of subsets of size \( n \) from \( \{1,2,\ldots,2n\} \).

The LHS is the sum of subsets of size \( k \) drawn from a set of size \( n \) - but each component in the sum is squared. Take the general term \( \binom{n}{k} \) and consider how one can choose a set of \( n \) elements from the bigger set \( \{1,2,\ldots,2n\} \) which contains \( k \) members of \( \{1,2,\ldots,n\} \).

We can choose \( k \) members from \( \{1,2,\ldots,n\} \) in \( \binom{n}{k} \) ways and having done that we can choose \( n-k \) members from \( \{n+1,n+2,\ldots,2n\} \) in \( \binom{n}{n-k} \) ways. Thus the total number of ways of making the choice is \( \binom{n}{k} \binom{n}{n-k} = \binom{n}{k}^2 \) because \( \binom{n}{n-k} = \binom{n}{k} \). Next add up this general term for \( k \)-subsets where \( k \) runs from 0 to \( n \) inclusive and you have picked up every subset of size \( n \) from \( 2n \) (which is the RHS). Summarising, on the RHS we focus on the bigger set \( \{1,2,\ldots,2n\} \) as a totality while on the LHS we split it up into \( \{1,2,\ldots,n\} \) and \( \{n+1,n+2,\ldots,2n\} \). We choose \( k \) elements from the first set and \( n-k \) from the second (so we have a subset of size \( n \) from the entire set) but we have to do that sum for \( 0 \leq k \leq n \).

The use of the binomial theorem revolves around the observation that \((x + y)^{2n} = (x + y)(x + y)^n \). What is the coefficient of \( x^a y^b \) of \((x + y)^n \)?

Focusing on \((x + y)^2 \) we have \((x + y)^{2n} = \sum_{k=0}^{n} \binom{2n}{k} x^k y^{2n-k} \) and the coefficient of \( x^a y^b \) occurs when \( n = k \) hence the coefficient is

\[
\binom{2n}{n}.
\]

So we have the RHS of the relationship. Now for the LHS we focus on \((x + y)^n \) as above. However, note that we have to fix \( k \) in the first sigma sum and for each such \( k \) we will have a corresponding term involving \( j \) (ie \( n - k \)), there being \( n+1 \) terms in all. Thus the coefficient of \( x^a y^b \) is \( \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}^2 = \sum_{k=0}^{n} \binom{n}{k}^2 \),

The coefficient of \( x^a y^b \) occurs in this latter expression when \( k + j = n \) hence the coefficient is \( \binom{n}{k} \binom{n}{n-k} = \binom{n}{k}^2 \) as above. However, next add

\[
\sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k} \binom{n}{n-k} x^{k+j} y^{n-(k+j)}
\]

21. Binary magic

Prove that \( C_0^n + C_1^n + C_2^n + C_3^n + C_4^n + \ldots = 2^{n-1} \) and \( C_1^n + C_3^n + C_5^n + C_7^n + \ldots = 2^{n-1} \) for all integers \( n \geq 1 \). Use the binomial theorem rather than induction.

Solution

The binomial theorem says that \( \sum_{k=0}^{n} C_k^n x^{n-k} y^k = (x + y)^n \) and letting \( x=y=1 \) we see that:

\( (1 + 1)^n = 2^n = \sum_{k=0}^{n} C_k^n = O_n + E_n \)

where \( O_n = C_1^n + C_3^n + C_5^n + \ldots \) and \( E_n = C_0^n + C_2^n + C_4^n + \ldots \) and \( (-1)^n C_n^n \)

Now \( E_n - O_n = C_0^n - C_1^n + C_2^n - C_3^n + \ldots = (-1)^n \) is 0

But we know from \textbf{Problem 8} that the RHS of this last equation = 0

So we have the following:

\( O_n + E_n = 2^n \)
\( E_n - O_n = 0 \)

Solving these two equations we see that \( E_n = 2^{n-1} = O_n \). We can see that this result holds for all \( n \) because the binomial theorem has been used along the way.

Another approach would be to observe that for any set with \( n \) members there is an even number of subsets (see \textbf{Problem 36}) which is \( 2^n \). There are \( n+1 \) coefficients which represent numbers of subsets of size \( k \) for \( 0 \leq k \leq n \) eg \( \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots \) etc. If \( n \) is odd the total such number of coefficients is even and thus the number of coefficients with even \( k \)-elements will equal the number of coefficients with odd \( k \)-elements.

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elements. Thus $2X = 2^0$ so $X = 2^{n-1}$ ie $C_n^0 + C_n^2 + C_n^4 + C_n^6 + \ldots = 2^{n-1}$ and $C_n^1 + C_n^3 + C_n^5 + C_n^7 + \ldots = 2^{n-1}$.

On the other hand if $n$ is even, then the total such number of coefficients is odd and but we can add one more coefficient $\binom{n}{n+1}=0$ (which doesn't change any sum) so that the number of coefficients with even k-elements will equal the number of coefficients with odd k-elements. Therefore $2X = 2^0$ so $X = 2^{n-1}$ in this case too. For instance take $n = 4$ then the coefficients would look like this: \[
\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}.
\]
In essence this is just an artifice to get the pairing to be consistent.

### 22. Simple but effective - you'll see why soon

(i) Guess a formula for $\frac{d^n(x^n)}{dx^n}$ for integers $n, m \geq 1$ and prove your guess by induction.

(ii) Guess a formula for $\frac{(2k)!}{k!}$ and prove it by induction.

#### Solution

These simple results will be used in problem 23.

(i) To make the notation a bit more efficient define $D^n = \frac{d^n}{dx^n}$ so that $\frac{d^n}{dx^n} = (\frac{d^{n-1}}{dx^{n-1}}) = D^{n-1}$

Then $D^n x^d = D^n[\frac{d}{dx} x^d] = 6D^n(\frac{x}{x^n}) = 6D(\frac{x^d}{x^n}) = 6.5. D(x^d) = 6.5.4. x^3$

Because the derivative is taken $n$ times the exponent of $x$ must drop by $n$. There are also $n$ factors out the front so you just count how much the index $m$ must drop to account for $n$ factors. This suggests that the formula is:

$D^n x^m = m(m-1)\ldots(m-(n-1))x^{m-n} = m(m-1)\ldots(m-n+1) x^{m-n}$

Note that there are $n$ factors running from $m$ to $m-(n-1) = m-n+1$

Does the formula work? In our example $m=6$ and $n=3$ so we get $6.5.4x^3$ as required. It looks like it is correct, however, once the power of $x$ is reduced to zero, any further derivatives will cause the answer to be zero because the derivative of a constant is zero.

Hence our formula becomes $D^n x^m = (m-1)\ldots(m-n+1) x^{m-n}$ for $n \geq m$

$= 0$ for $n > m$

When $n=m=1$ we have $Dx = 1. x^0 = 1$. So the formula is true for $m=n=1$

Assume the formula is true for all $n, m$ such that $n \geq m$. Then $D^{m+1} x^{n+1} = D^n(Dx^{n+1}) = (m+1)D^n x^m = (m+1) (m-m-1)\ldots(m+n+1) x^{m+n+1}$ using the induction hypothesis. Hence the formula is true for $n+1, m+1$ since it is true for $n=m=1$

Note that $D^n x^n = n!$ which is a subcase of the more general case. You can prove that assertion inductively very quickly since the proposition is clearly true for $n=1$ and $D^{n+1} x^{n+1} = D^n(Dx^{n+1}) = (n+1)D^n x^n = (n+1) n! = (n+1)!$. Hence the formula is true for $n+1$ and thus true for all $n$.

This is a handy result when you come to manipulate series of polynomial functions.

(ii) To guess the answer simply write it all out and look for some obvious symmetry.

\[
\frac{(2k)!}{k!} = \frac{(2k)(2k-1)(2k-2)(2k-3)\ldots3\cdot2\cdot1}{k(k-1)(k-2)\ldots3\cdot2\cdot1}
\]

\[
= \frac{2\cdot3\cdot4\cdot\ldots(2k-1)(2k-2)(2k-3)\ldots3\cdot2\cdot1}{k(k-1)(k-2)\ldots3\cdot2\cdot1}
\]

\[
= 2^{n+1} (2k-1)(2k-3)\ldots3\cdot1
\]

Now there are $2k$ terms in $(2k)!$ and $k$ terms in $k!$. Thus each of the $k$ terms in $k!$ will cancel with a term in the numerator so a guess for "some power" is $k$.

Hence the assertion is that $\frac{(2k)!}{k!} = 2^k(2k-1)(2k-3)\ldots3\cdot1$

This formula is true for $n = 1$ since $\frac{2^1}{1!} = 2$ and RHS = $2^1(2-1) = 2$. Assume the formula is true for any $n$. 

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Then \(\frac{(2n+2)!}{(n+1)!} = \frac{(2n+2)(2n+1)(2n)!}{(n+1)n!} = \frac{2(2n+1)2^n(2n-1)(2n-3)...3\cdot 2\cdot 1}{n!}\) using the induction hypothesis

\[= 2^{n+1}(2n+1)(2n-1)(2n-3)\ldots 3 \cdot 2 \cdot 1\] so the formula is true for \(n+1\) and hence true for all \(n\).

### 23. An application of induction to Legendre polynomials

Legendre polynomials arise in the context of several problems in mathematical physics (e.g., celestial mechanics). An important property of Legendre polynomials is that they are orthogonal on the interval \([-1, 1]\). The concept of orthogonality of functions is important in the context of many aspects of mathematical physics such as Fourier theory, Hilbert space theory and so on. In the context of Legendre polynomials the orthogonality condition is expressed as \(\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0\) when \(n \neq m\). This is the analogue of orthogonality of normal vectors.

Legendre polynomials are defined as follows (see Courant and Hilbert, "Methods of Mathematical Physics" Volume 1, Wiley Classics Library, 1989, pages 83-84). This is a very famous textbook and rightly so. The Hilbert is none other than the David Hilbert and Courant is none other than Richard Courant who went to the US from Germany in the 1930s to establish the Courant Institute. The purpose of this exercise is to take some important results and show how the professionals do it and also to show what they leave out!

This is serious applied maths - enter at your own risk. You will need to review integration by parts because it is fundamental to this problem.

The \(n\)th Legendre polynomial \(y(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n}\) satisfies the second order linear homogeneous differential equation \((x^2 - 1)y'' + 2xy' - n(n+1)y = 0\). Why bother with what follows below? If you do any applied maths, especially differential equations, you will come across Legendre, Laguerre, Hermite, Jacobi and Chebyshev polynomials in the context of several famous applications. The ability to manipulate the often quite complicated series that arise is a very useful skill. The same goes for Fourier Theory. Inductive arguments are often implicitly used in the simplification process.

\(P_0(x) = 1, \quad P_1(x) = \frac{1}{2} \frac{d(x^2 - 1)}{dx}\) \(n = 1, 2, \ldots\).

The authors then state the following:

\[
P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} (2k)! (2n)! 2^{2k - n} \sum_{k=0}^{n} \binom{n}{k} (2k - 1) 5^k (2k - 0)!^2 2^{k - n} x^{2k - n}
\]

They note that "the terms in negative powers are to be omitted; this comes about automatically of we put \((-1)^n = \infty\) for all positive integers \(n\). What they are doing here is simply a bit of formalism - you can run the \(k\) index from 0 to \(n\) and along the way you will get negative exponents but if the denominator is \(\infty\) those terms go. It just means that you can play with the summation in that form and not worry about the details. Of course, you will not get negative exponents from \((x^2 - 1)^n\) when you differentiate - the highest power is \(2n\) and when you differentiate \(n\) times you won't get a negative power. Note that \(P_0(x)\) is a polynomial of degree \(n\) as can be seen from the exponent of \(x^{2k - n}\) when \(k = n\).

(i) Prove these results by induction. Hint: You may want to use the result you obtained in Problem 22.

Verify the formula for a few Legendre polynomials, namely:

\[P_1(x) = x, \quad P_2(x) = \frac{3}{2} x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2} x^3 - \frac{3}{2} x\]

(ii) Prove that the \(P_n(x)\) form an orthogonal system. Courant and Hilbert offer the following proof. They denote \((x^2 - 1)^n\) by \(u_n(x)\) and they claim that for every non-negative integer \(m < n\):

\[
\int_{-1}^{1} P_n(x) x^m \, dx = \frac{1}{2^n n!} \int_{-1}^{1} u_n^{(m)}(x) \, dx = 0 \quad \text{(note that } u_n^{(m)}(x) = D^m u_n(x)\text{)}
\]

They say that this "may be demonstrated if we remove the factor \(x^m\) by repeated partial integration and note that the derivatives of \(u_n(x)\) up to the \((n-1)\)-st vanish at the limits of the integral of integration. It follows that \(\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0\)".

Fill in the details.

(iii) To prove that the \(P_n(x)\) form an orthogonal system we have to show that they are appropriately normalised. The authors say that "in order to obtain the necessary normalization factors we now compute \(\int_{-1}^{1} (u_n(x))^2 \, dx\) by repeated partial integration:

\[
\int_{-1}^{1} u_n^{(0)}(x) u_n^{(0)}(x) \, dx = - \int_{-1}^{1} u_n^{(n-1)}(x) u_n^{(n+1)}(x) \, dx = \int_{-1}^{1} u_n^{(n-2)}(x) u_n^{(n+2)}(x) \, dx = \ldots = (-1)^n \int_{-1}^{1} u_n(x) u_n^{(2n)}(x) \, dx
\]

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Now \( D_1 \int_0^1 (1-x)^n (1+x)^n \, dx \)

= \( (2n)! \int_0^1 (1-x)^n (1+x)^n \, dx \)

Now \( D_1 \int_0^1 (1-x)^n (1+x)^n \, dx = \frac{n}{n+1} D_1 \int_0^1 (1-x)^{n-1} (1+x)^n \, dx = \ldots = \frac{n(n-1)}{(n+1)(n+2) \ldots (2n)} D_1 \int_0^1 (1-x)^{2n} \, dx \)

= \( \frac{(2n)!}{(2n+1)!} 2^{2n+1} \) and therefore \( D_1 \int_0^1 P_n(x) \, dx \) = \( \frac{2}{2n+1} \). The desired normalized polynomials are therefore \( \varphi_0(x) = \sqrt{\frac{2n+1}{2}} P_0(x) \) * 

This means that \( D_1 \int_0^1 \varphi_0(x) \, dx = 1 \)

Now fill in the details for each of the steps given above.

(iii) By starting with the differential equation \((x^2-1)y' + 2xy - n(n+1) y = 0\) use integration by parts to show that \( D_1 \int_0^1 P_n(x) P_m(x) \, dx = 0 \). This is a non-inductive proof.

**Solutions**

(i) To get the required expression for \( P_n(x) \) you start with the binomial expansion for \((x^2 - 1)^n\) which is \( \sum_{k=0}^{n} \binom{n}{k} x^{2k} (-1)^{n-k} \) which you get by taking \( a = x^2 \) and \( b = -1 \) in the binomial expansion of \((a + b)^n\). The \( n^{th} \) derivative of \((x^2 - 1)^n\) is \( D^n \left( \sum_{k=0}^{n} \binom{n}{k} x^{2k} (-1)^{n-k} \right) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} D^n [x^{2k}] \) using the linearity of the derivative operator \( D = \frac{d}{dx} \).

**Problem 22(i)** enables us to work out \( D^n \left[ x^{2k} \right] = (2k)(2k-1) \ldots (2k - n + 1) x^{2k-n} \)

Thus we have \( P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (2k)(2k-1) \ldots (2k - n + 1) x^{2k-n} \)

We now just need to do a bit of simplification.

\[ P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{n} (-1)^{n-k} n \binom{n}{k} (2k)(2k-1) \ldots (2k - n + 1) x^{2k-n} \]

\[ = \sum_{k=0}^{n} (-1)^{n-k} \frac{2k(2k-1) \ldots (2k - n + 1) x^{2k-n}}{(n-k)!} \]

\[ = \sum_{k=0}^{n} (-1)^{n-k} \frac{(2k)! x^{2k-n}}{2^n (n-k)! (2k - n)!} \]

\[ = \sum_{k=0}^{n} (-1)^{n-k} \frac{k x^{2k-n}}{2^n (n-k)! (2k - n)!} \]

\[ = \sum_{k=0}^{n} (-1)^{n-k} \frac{\frac{1}{2} \frac{5}{2} \ldots \frac{(2k-1)}{2} \frac{1}{2} x^{2k-n}}{2^n (n-k)! (2k - n)!} \]

As a demonstration of the formula: \( P_3(x) = \sum_{k=0}^{3} (-1)^{3-k} \frac{\frac{1}{2} \frac{5}{2} \ldots \frac{(2k-1)}{2} \frac{1}{2} x^{2k-3}}{2^n (n-k)! (2k - n)!} \) (note how the negative powers of \( x \) are ignored)

(ii) We have to \( D_1 \int_0^1 P_n(x) x^m \, dx = \frac{1}{2^n n!} \sum_{k=0}^{n} (-1)^{n-k} \frac{\frac{1}{2} \frac{5}{2} \ldots \frac{(2k-1)}{2} \frac{1}{2} x^{2k-3}}{2^n (n-k)! (2k - n)!} \)

Now this : \( x^n D^{n-1} u_n(x) \left( \frac{1}{1} - \int_0^x m D^{n-1} u_n(x) x^{m-1} \, dx \right) \)

To establish this do an integration by parts as follows: \( \int_0^1 P_n(x) x^m \, dx = \left[ \frac{D^n u_n(x) x^m}{D x} \right]_0^1 - \int_0^1 D^n u_n(x) x^m \, dx = \left[ \frac{D^n u_n(x) x^m}{D x} \right]_0^1 - \int_0^1 m D^{n-1} u_n(x) x^{m-1} \, dx \)

This can be seen either by noting that when \((x^2 - 1)^n\) is differentiated you get a common factor of \((x^2 - 1)\) throughout (just try it) and when \( x = 1 \) or \(-1\) this is zero. It is critical that only
derivatives of order up to n-1 are taken - if you go to order n you will get a constant and \( x^n D^{n-1} u_n(x) \left|^{1}_{1} \right. \) will not generally be zero.

Leibniz’s rule also helps to see this when written in the form \( D^{n-1} u_{n-1} u_1 = D^{n-1}( (x^2 - 1)^{n-1} (x_2 - 1) ) = \sum_{k=0}^{n-1} \binom{n-1}{k} D^{n-1-k} u_{n-1} D^k u_1 \).

Note that \( D^k u_1 = u_1 \) and \( u_1 = 0 \) at \( x = 1 \) and \( -1 \). It is clear that \( D^k u_1 = D^k (x^2 - 1) = 0 \) for \( k \geq 3 \) so one only needs to focus on \( D^{n-k} u_{n-1} \) in the product for \( k = 1 \) and 2 evaluated at \( x = 1 \) and \(-1\).

In each such term there is a factor \((x^2 - 1)\) which is zero at \( x = 1 \) and \(-1\).

Thus after one integration by parts we have: \( \int_1^1 P_n(x) x^m \, dx = -\int_1^1 m D^{n-1} u_n(x) x^{m-1} \, dx \). This is a classic inductive style of approach because we do the same again to get:

\[
\int_1^1 m D^{n-1} u_n(x) x^{m-1} \, dx = \int_1^1 (m-1) D^{n-2} u_n(x) x^{m-2} \, dx.
\]

After \( m \) iterations the integral will reduce to something of the form \((-1)^m m! \int_1^{1} D^{n} u_n(x) \, dx = m! \int_1^{1} D^{(x^2 - 1)^{n}} \, dx = m! \left( (x^2 - 1)^{n} \right) |^{1}_{1} = 0 \) (you don’t need to get precious about the multiplicative factor which is just window dressing - the real focus is the final form of the integrand)

(iii) Starting with \( \int_1^{1} u_n^{(n)}(x) u_n^{(0)}(x) \, dx \) we integrate by parts remembering that the derivatives up to order \( n -1 \) evaluated at \( 1 \) and \(-1\) are zero:

\[
\int_1^{1} u_n^{(n)}(x) u_n^{(0)}(x) \, dx = \int_1^{1} D u_n^{(n-1)}(x) u_n^{(0)}(x) \, dx = \left[ u_n^{(n-1)}(x) u_n^{(0)}(x) \right] |^{1}_{1} - \int_1^{1} u_n^{(n-1)}(x) u_n^{(1)}(x) \, dx = -\int_1^{1} u_n^{(n-1)}(x) u_n^{(1)}(x) \, dx.
\]

Thus after one integration by parts we get a factor of \(-1\) out front and the derivative order of one factor drops by one while the other increases by one. If we do \( n \) integrations by parts we will get:

\[
(-1)^n \int_1^{1} u_n^{(n)}(x) u_n^{(0)}(x) \, dx = (-1)^n \int_1^{1} u_n^{(n)}(x) \, dx \quad \text{since } u_n^{(n)}(x) = u_n^{(0)}(x) \text{ ie no derivative is taken. Note that each time the integration by parts is done the term } \left[ u_n^{(n-1)}(x) u_n^{(0)}(x) \right] |^{1}_{1} \text{ and its "descendants" vanishes because of the property established in (ii).}
\]

Now \( (-1)^n \int_1^{1} u_n^{(n)}(x) u_n^{(0)}(x) \, dx = (-1)^n \int_1^{1} (x^2 - 1)^n \, dx \) since \( u_n^{(n)}(x) = u_n^{(0)}(x) \) is no derivative is taken.

But \( u_n(x) = (x^2 - 1)^n \) is a polynomial whose highest power is \( x^{2n} \) and we know from Problem 22(i) that \( D^{2n} x^{2n} = (2n)! \) which is what we want. All the other terms are zero when differentiated \( 2n \) times.

So \( \int_1^{1} u_n^{(0)}(x) u_n^{(0)}(x) \, dx = \int_1^{1} (1 - x)^n (1 + x)^n u_n^{(0)}(x) \, dx = (2n)! \int_1^{1} (1 - x)^n (1 + x)^n \, dx \) as claimed by the authors.

The next thing to show is that \( \int_1^{1} (1 - x)^n (1 + x)^n \, dx = \frac{n!}{(n+1)(n+2)...(2n)} \int_1^{1} (1 - x)^n \, dx \)

Once again integration by parts is needed.

\[
\int_1^{1} (1 - x)^n (1 + x)^n \, dx = \left[ (1 - x)^n (1 + x)^n \right] |^{1}_{1} + \int_1^{1} (1 - x)^n (1 + x)^n \, dx = \frac{n!}{(n+1)(n+2)...(2n)} \int_1^{1} (1 - x)^n \, dx
\]

Thus the leading term in the integration by parts is zero at each iteration and the exponent of the \( (1 - x)^n \) factor increases by one each iteration while that of the \( (1 + x)^n \) decreases by one and there is a decrease each time due to the differentiation of that term. The decreasing factors in the numerator are associated with the \( (1 - x)^n \) factor while increasing factors in the denominator are associated with the \( (1 + x)^n \) factor. Thus after \( n \) partial integrations you get

\[
\int_1^{1} (1 - x)^n (1 + x)^n \, dx = \frac{n!}{(n+1)(n+2)...(2n)} \int_1^{1} (1 + x)^n \, dx
\]

Recall that \( P_n(x) \) is defined as \( \frac{1}{2^n n!} u_n^{(0)}(x) \) so from \( \int_1^{1} u_n^{(0)}(x) u_n^{(0)}(x) \, dx = (2n)! \int_1^{1} (1 - x)^n (1 + x)^n \, dx = \frac{(2n)! (n!)^2}{(2n)!} \) we have that:

\[
\int_1^{1} P_n(x) \, dx = \left( \frac{1}{2^n n!} \right)^2 \int_1^{1} u_n^{(0)}(x) u_n^{(0)}(x) \, dx = \left( \frac{1}{2^n n!} \right)^2 \int_1^{1} (1 - x)^n (1 + x)^n \, dx = \left( \frac{1}{2^n n!} \right)^2 \frac{(2n)! (n!)^2}{(2n)!} \frac{1}{2n+1}
\]
Induction step:

and we have already proved the proposition for

(iii) Integration by parts is the big hint here. The differential equation

Then:

This chain of equivalences shows that

Note that this whole result can be generalised as follows: \( \int_0^\beta P_\alpha(x) \, d\alpha = 0 \) if \( m \neq n \) and \( \frac{\beta-\alpha}{2n+1} \) if \( m = n \).

(iii) Integration by parts is the big hint here. The differential equation \((x^2-1)\gamma'' + 2xy\gamma' - n(n+1)\gamma = 0\) can be written as:

Now let \( y = P_\alpha(x) \) and then multiply by \( P_\beta(x) \) and integrate between \( x = -1 \) and \( x = 1 \) to get:

\[
\left\{ \int_{-1}^1 \frac{d}{dx} \left((x^2-1)P_\alpha(x)\right) P_\beta(x) \, dx \right\} \, d\alpha = \int_{-1}^1 n(n+1) P_\alpha(x) P_\beta(x) \, dx
\]

\[
(x^2-1) P_\alpha(x) P_\beta(x) \bigg|_{-1}^1 - \int_{-1}^1 (x^2-1) P_\alpha(x) P_\beta(x) \, dx = \int_{-1}^1 n(n+1) P_\alpha(x) P_\beta(x) \, dx
\]

Finally:

\[
-\int_{-1}^1 (x^2-1) P_\alpha(x) P_\beta(x) \, dx = \int_{-1}^1 n(n+1) P_\alpha(x) P_\beta(x) \, dx
\]

Now let \( y = P_\alpha(x) \) in the differential equation and multiply by \( P_\beta(x) \). This gives rise to a symmetrical result namely:

\[
-\int_{-1}^1 (x^2-1) P_\alpha(x) P_\beta(x) \, dx = \int_{-1}^1 (m(m+1)) P_\alpha(x) P_\beta(x) \, dx
\]

Thus:

\[
\int_{-1}^1 (m(m+1)) P_\alpha(x) P_\beta(x) \, dx = \int_{-1}^1 n(n+1) P_\alpha(x) P_\beta(x) \, dx
\]

which means that:

\[
\int_{-1}^1 P_\alpha(x) P_\beta(x) \, dx = 0 \quad \text{if} \quad m \neq n
\]

24. The Cauchy-Schwarz formula

Without any doubt the Cauchy-Schwarz formula is one of the most important mathematical relationships that you can know. It arises in every area of mathematics and has a vast number of applications.

What is the Cauchy-Schwarz formula?

For real \( a_i \) and \( b_i \), \( i = 1, 2, \ldots, n \):

\[
\sum_{i=1}^n a_i b_i \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}
\]

Prove the Cauchy-Schwarz formula for integral \( n \geq 1 \).

Solution

When \( n = 1 \) the formula is trivially true and does not give any insight into the more general proof. So try \( n = 2 \). Then Monsieur Cauchy asserts that:

\[
\left(\sum_{i=1}^n a_i b_i\right)^2 = (a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)
\]

iff \( a_1 b_1^2 + 2 a_1 b_2 a_2 b_2 \leq a_1^2 b_1^2 + a_2^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_1^2 + a_2^2 b_2^2
\]

iff \( 0 \leq (a_1 b_2)^2 - 2 a_1 b_2 a_2 b_2 + (a_2 b_1)^2 = (a_1 b_2 - a_2 b_1)^2
\]

This chain of equivalences shows that

\[
(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)
\]

is indeed true and hence the formula is true for \( n = 2 \).

Induction step:

\[
\sum_{i=1}^{n+1} a_i b_i = \sum_{i=1}^n a_i b_i + a_{n+1} b_{n+1} \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} + a_{n+1} b_{n+1}
\]

Now let \( \alpha = \sqrt{\sum_{i=1}^n a_i^2} \) and \( \beta = \sqrt{\sum_{i=1}^n b_i^2} \)

Then:

\[
\sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2} + a_{n+1} b_{n+1} = \alpha \beta + a_{n+1} b_{n+1}
\]

and we have already proved the proposition for \( n = 2 \) so we can apply it to \( \alpha \beta + a_{n+1} b_{n+1} \) – this is the critical insight that allows the
problem to be solved.

So \(a_{n+1} b_{n+1} \leq (a^2 + a_{n+1}^2)^{\frac{1}{2}} (b^2 + b_{n+1}^2)^{\frac{1}{2}}\)

Hence \(\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 + a_{n+1} b_{n+1} \leq (\sum_{i=1}^{n} a_i^2 + a_{n+1}^2)^{\frac{1}{2}} (\sum_{i=1}^{n} b_i^2 + b_{n+1}^2)^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{n+1} a_i^2} \sqrt{\sum_{i=1}^{n+1} b_i^2}\)

So the formula is true for \(n + 1\) and hence is proved by induction

25. A French interlude

De Moivre’s Theorem: \((\cos x + i \sin x)^n = \cos (nx) + i \sin (nx)\) where \(x\) is real.

Prove this theorem for \(n \in \mathbb{Z}\) ie \(n\) integral (positive, negative and zero).

Solution

The formula is obviously true for \(n=1\). To solve this problem you have to recall that \(\cos (x + y) = \cos x \cos y - \sin x \sin y\) and \(\sin (x + y) = \sin x \cos y + \cos x \sin y\)

Assume the formula is true for any positive integer \(n\). Then

\((\cos x + i \sin x)^{n+1} = [\cos (nx) + i \sin (nx)] (\cos x + i \sin x)\)

\[= \cos(nx) \cos x + i \cos(nx) \sin x + i \sin (nx) \cos x - \sin (nx) \sin x\]

\[= \cos(nx) \cos x \cdot \sin (nx) \sin x + i [\cos(nx) \sin x + \sin (nx) \cos x]\]

\[= \cos [(n+1)x] + i \sin [(n+1)x]\]

So the formula is true for \(n+1\) and hence for all \(n \geq 1\)

Now consider negative \(n\)

When \(n = -1\), \((\cos x + i \sin x)^{-1} = \frac{1}{\cos x + i \sin x}\)

\[= \frac{1}{\cos x + i \sin x} \frac{\cos x - i \sin x}{\cos x - i \sin x}\]

\[= \frac{\cos x - i \sin x}{\cos^2 x + \sin^2 x}\]

\[= \cos x - i \sin x\]

\[= \cos (-x) + i \sin (-x)\] (cos is an even function while sin is an odd function)

So the formula is true for \(n = -1\)

Assume the formula is true for any integer \(n = -k\) where \(k\) is a positive integer.

Then \((\cos x + i \sin x)^{-(k+1)} = [\cos x + i \sin x]^{-k} (\cos x + i \sin x)^{-1}\)

\[= [\cos (-kx) + i \sin (-kx)] [\cos (-x) + i \sin (-x)]\] (using the induction hypothesis and the fact that we proved the formula for \(n = -1\))

\[= \cos (-kx) \cos (-x) - \sin (-kx) \sin (-x) + i [\sin (-kx) \cos (-x) + \cos (-kx) \sin (-x)]\]

\[= \cos [-(-k+1)x] + i \sin [-(-k+1)x]\]

The formula is also true for \(n=0\) since \((\cos x + i \sin x)^0 = 1 = \cos 0 + i \sin 0\).

So the formula is true for \(n = -k+1\) and since it is true for \(n=-k\) and \(n=-1\) it is true for all negative integers.

26. Another binomial example

\(1+2+4+\ldots+2^{n-1} = 2^n - 1\)

Prove this formula for integral \(n \geq 1\)
Solution

When \( n=1 \), LHS = \( 2^{1-1} = 2^0 = 1 \)
RHS = \( 2^1 - 1 = 1 \)
So the formula is true for \( n=1 \)

Assume the formula is true for any \( n \). Then:

\[
1 + 2 + 4 + \ldots + 2^{n-1} + 2^n = 2^n - 1 + 2^n \text{ using the induction hypothesis}
\]

\[
= 2^{n+1} - 1
\]
Thus the formula is true for \( n+1 \) and hence is true for all \( n \geq 1 \).

27. Factors

(i) Prove that \( 5^n + 2(11^n) \) is a multiple of 3 for all positive integers \( n \)

Solution

When \( n=1 \) the formula yields \( 5 + 2 \cdot 11 = 27 \) which is divisible by 3. Assume the formula is true for any \( n \) ie \( 5^n + 2(11^n) = 3q \) where \( q \) is a positive integer. Then:

\[
5^{n+1} + 2 \cdot 11^{n+1} = 5 \cdot 5^n + 2 \cdot 11 \cdot 11^n
\]

\[
= 5 \cdot 5^n + 5 \cdot 2 \cdot 11^n - 5 \cdot 2 \cdot 11^n + 2 \cdot 11 \cdot 11^n
\]

\[
= 5 \left(5^n + 2 \cdot 11^n \right) - 10 \cdot 11^n + 22 \cdot 11^n
\]

\[
= 15q + 12 \cdot 11^n
\]

\[
= 3 \left(5q + 4 \cdot 11^n\right)
\]

\[
= 3q' \text{ where } q' = 5q + 4 \cdot 11^n
\]

Hence \( 5^{n+1} + 2 \cdot 11^{n+1} \) is divisible by 3 and the proposition is proved by induction.

(ii) Prove that 17 divides \( 7^n - 1 \) for integral \( n \) \( n \geq 1 \) two ways - first by induction and then by using the binomial theorem.

Solution

The formula clearly holds for \( n = 1 \) since \( 6 \mid 7 - 1 \). Assume that the formula holds for any \( n \). Then:

\[
7^{n+1} - 1 = 7 \cdot 7^n - 1 = 7 \left(7^n - 1 \right) + 6
\]

Using the induction hypothesis \( 6 \mid 7^n - 1 \so 7^n - 1 = 6r \)

Thus \( 7^{n+1} - 1 = 7 \cdot 6r + 6 \) and this is clearly divisible by 6 so the formula is true for \( n+1 \) and hence true for all \( n \) by induction.

The binomial theorem tells us that \((1 + 6)^n = \sum_{k=0}^{n} \binom{n}{k} 6^{n-k} = 1 + \sum_{k=1}^{n} \binom{n}{k} 6^{n-k} \)

Thus \( 7^n - 1 = \sum_{k=1}^{n} \binom{n}{k} 6^{n-k} \) which is clearly divisible by 6

(iii) Prove by induction that \( 17 \mid 3 \cdot 5^{2n+1} + 2^{3n+1} \) for integral \( n \geq 1 \)

Solution

The formula is true for \( n = 1 \) since \( 3 \cdot 5^3 + 2^4 = 391 = 17 \cdot 23 \)

Assume as the induction hypothesis that \( 17 \mid 3 \cdot 5^{2n+1} + 2^{3n+1} \) ie \( 3 \cdot 5^{2n+1} + 2^{3n+1} = 17r \)
Now for $n+1$ we have: $3.5^{2n+3} + 2^{3n+4} = 25.3.5^{2n+1} + 8.2^{3n+1}$

$= 25 \times (3.5^{2n+1} + 2^{3n+1}) - 17 \times 2^{3n+1}$

$= 25.17r - 17.2^{3n+1}$ using the induction hypothesis

Hence $17 | 3.5^{2n+3} + 2^{3n+4}$ and the truth of the formula is established by induction.

28. “Where’s Wally?” or spot the error

The relationship between the arithmetic mean (AM) and geometric mean (GM) is as follows:

For every sequence of non-negative real numbers $a_1, a_2, \ldots, a_n$ we have:

$$(a_1 a_2 \ldots a_n)^\frac{1}{n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

Here is a “proof” of the theorem. Explain where and why the proof fails (each line is numbered for this purpose). It is like that children’s book called “Where’s Wally?” which invites children to concentrate on spotting little Wally among a lot of visual clutter.

"Proof:

The formula is true for $n=1$. (1)

Assume the formula is true for any $n$. (2)

Now $$(a_1 a_2 \ldots a_n a_{n+1})^\frac{1}{n+1} = (a_1 a_2 \ldots a_n)^\frac{1}{n} a_{n+1}^\frac{1}{n+1}$$ (3)

$$(a_1 a_2 \ldots a_n)^\frac{1}{n} a_{n+1}^\frac{1}{n+1} = (a_1 a_2 \ldots a_n)^\frac{1}{n} a_{n+1}^\frac{1}{n+1}$$ (4)

$$(a_1 a_2 \ldots a_n)^\frac{1}{n} a_{n+1}^\frac{1}{n+1} = (a_1 a_2 \ldots a_n)^\frac{1}{n} a_{n+1}^\frac{1}{n+1}$$ using the induction hypothesis (5)

But $a_{n+1}^\frac{1}{n+1} = (a_{n+1} \cdot 1 \ldots 1)^\frac{1}{n+1}$ (6) (there are $(n-1)$ “1’s” in the brackets)

Using the induction hypothesis again: $$(a_{n+1} \cdot 1 \ldots 1)^\frac{1}{n+1} = \frac{(a_{n+1} + 1 \ldots 1)}{n+1}$$ (7)

Hence we have that $$(a_1 a_2 \ldots a_n a_{n+1})^\frac{1}{n+1} = (a_1 a_2 \ldots a_n)^\frac{1}{n} a_{n+1}^\frac{1}{n+1}$$ (8)

$$(\frac{a_1 + a_2 + \cdots + a_n}{n})^\frac{1}{n} = (\frac{a_1 + a_2 + \cdots + a_n}{n})^\frac{1}{n}$$ (9)

$$(\frac{a_1 + a_2 + \cdots + a_n}{n})^\frac{1}{n} = (\frac{a_1 + a_2 + \cdots + a_n}{n+1}) (a_{n+1} + 1)$$ (10)

$$(\frac{a_1 + a_2 + \cdots + a_n}{n+1}) (a_{n+1} + 1)$$ (11)

The chain of inequalities shows that $(a_1 a_2 \ldots a_n a_{n+1})^\frac{1}{n+1} = (\frac{a_1 + a_2 + \cdots + a_n}{n+1})$ hence the formula is true for $n+1$ and by the principle of induction is true for all $n$. (12) QED!"

Solution

"Proof:

The formula is true for $n=1$. (1) This is correct

Assume the formula is true for any $n$. (2) This is correct

Now $$(a_1 a_2 \ldots a_n a_{n+1})^\frac{1}{n+1} = (a_1 a_2 \ldots a_n)^\frac{1}{n} a_{n+1}^\frac{1}{n+1}$$ (3) This is correct since $(ab)^\frac{1}{2} = a^\frac{1}{2} b^\frac{1}{2}$

$$(a_1 a_2 \ldots a_n)^\frac{1}{n} a_{n+1}^\frac{1}{n+1} = (a_1 a_2 \ldots a_n)^\frac{1}{n} a_{n+1}^\frac{1}{n+1}$$ (4) This is correct. If $0 < x < y$ then $a^x < a^y$ where $a > 0$. Note that $n < n+1$ \Rightarrow
\[\frac{1}{n} \leq \frac{1}{n+1}\]

\[(a_1 a_2 \ldots a_n)^\frac{1}{n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}\]  using the induction hypothesis \hspace{1cm} (5) \hspace{1cm} This is correct.

But \[a_{n+1} \leq (a_{n+1}, \ldots, 1)\] \hspace{1cm} (there are (n-1) “1"s” in the brackets) \hspace{1cm} (6)

There are 2 steps in this line. It is true that \[a_{n+1} \leq (a_{n+1}, \ldots, 1)\] since the padding with ones changes nothing, and since \[\frac{1}{n+1} < \frac{1}{n}\] it follows that \[(a_{n+1}, \ldots, 1)^\frac{1}{n+1} \leq (a_{n+1}, \ldots, 1)^\frac{1}{n}\] . So this line is correct (the equality would hold when \(a_{n+1} = 1\) or 0)

Using the induction hypothesis again: \[(a_{n+1}, \ldots, 1)^\frac{1}{n} \leq \frac{(a_{n+1}, \ldots, n+1)}{n}\] \hspace{1cm} (7) \hspace{1cm} This is correct since there are \(n\) terms in the bracketed term on the LHS the induction hypothesis guarantees the RHS (there are (n-1) 1’s as noted above and \(n\) terms in all)

Hence we have that \[(a_1 a_2 \ldots a_n a_{n+1})^\frac{1}{n+1} \leq \frac{(a_1 a_2 + \ldots + a_n a_{n+1} + 1)}{n+1}\] \hspace{1cm} (8) \hspace{1cm} This is correct since the above chain of implications holds.

\[
\frac{(\frac{a_1 a_2 + \ldots + a_n}{n})}{(\frac{a_1 a_2 + \ldots + a_n}{n}) (\frac{a_{n+1} + 1}{n})}
\]

(9) \hspace{1cm} This is correct since the numerator is increased - remember that \(a_{n+1} \geq 0\)

\[
\frac{\frac{(\frac{a_1 a_2 + \ldots + a_n}{n})}{(\frac{a_1 a_2 + \ldots + a_n}{n}) (\frac{a_{n+1} + 1}{n})}}{\frac{\frac{\frac{a_1 + a_2 + \ldots + a_n}{n}}{\frac{a_1 + a_2 + \ldots + a_n}{n}}}{\frac{a_{n+1} + 1}{n}}}
\]

but \[\frac{1}{n+1} \neq \frac{1}{n}\].

The "proof" is flawed and there is no need to go any further.

29. Now for the real proof of the AM-GM theorem

For every sequence of non-negative real numbers \(a_1, a_2, \ldots, a_n\) we have:

\[(a_1 a_2 \ldots a_n)^\frac{1}{n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}\]

Prove this by backward induction.

Hint: you may find the following inequality useful (after you’ve proved it of course)

\[
\sqrt{xy} \leq \frac{x}{2} + \frac{y}{2} \ni \text{for all non-negative } x \neq y
\]

Solution

The hint gives us the formula when n=2 and is an example where starting at something other than the trivially true is more evocative of the solution. To prove \(\sqrt{xy} \leq \frac{x}{2} + \frac{y}{2}\) start with this embarrassingly trivial statement: \((x - y)^2 = 0\) (where x and y are non-negative and unequal).

This leads inexorably to: \(x^2 - 2xy + y^2 \geq 0\) \hspace{1cm} ie \hspace{1cm} 2xy \hspace{1cm} x^2 + y^2

Let \(x = \sqrt{u}\) and \(y = \sqrt{v}\) (we can do this because x and y are assumed non-negative ie they are the root of some non-negative number)

Then \(2\sqrt{uv} = u + v\)

ie \(\sqrt{uv} = \frac{u}{2} + \frac{v}{2}\) which is what we wanted to show

At this stage know that \[(a_1 a_2)^\frac{1}{2} \leq \frac{a_1 + a_2}{2}\]

Let’s see what happens if we go to: \(a_1 a_2 a_3 a_4)^\frac{1}{4} = \left(\left(a_1 a_2\right)^\frac{1}{2} \right) \left(a_3 a_4\right)^\frac{1}{2}\) \hspace{1cm} (make sure you understand why this is right - it is because \(x^a y^b = x^{ab} y^{2b}\))

This looks like it might work: \(\left(a_1 a_2\right)^\frac{1}{2} \left(a_3 a_4\right)^\frac{1}{2}\)

\[\frac{a_1 + a_2}{2} + \frac{a_3 + a_4}{2} + \frac{a_1 + a_2 + a_3 + a_4}{4}\] using the fact that \(a_1 a_2)^\frac{1}{2} \frac{a_1 + a_2}{2}\]
When we do this again we get:  \((a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8)^{\frac{1}{7}} = \left( \left( a_1 a_2 a_3 a_4 \right)^{\frac{1}{7}} \left( a_5 a_6 a_7 a_8 \right)^{\frac{1}{7}} \right)^{\frac{4}{7}} \) using the same logic as above.

This suggests that  \((a_1 a_2 \ldots a_{2^n})^\frac{1}{2^n} \leq \frac{a_1 + a_2 + \ldots + a_{2^n}}{2^n}\) for \(k = 1\) (This is our claim - call it \(P(k)\)).

Now prove the claim by induction. We know that \(P(1)\) is true. Assume it is true for an arbitrary positive integer \(k\).

Then  \(P(k+1) = (a_1 a_2 \ldots a_{2^{k+1}})^{\frac{1}{2^{k+1}}} = (a_1 a_2 \ldots a_{2^k} a_{2^k+1} a_{2^k+2} \ldots a_{2^{k+1}})^{\frac{1}{2^{k+1}}} = ((a_1 a_2 \ldots a_{2^k} a_{2^k+1})^\frac{1}{2^k} (a_{2^k+2} \ldots a_{2^{k+1}})^{\frac{1}{2^{k+1}}} \right)^\frac{1}{2^{k+1}}\) if you don’t understand this step just look at how we split up the case where \(n = 2^3 = 8\)

But  \((a_1 a_2 \ldots a_{2^k} a_{2^k+1}) (a_{2^k+2} \ldots a_{2^{k+1}})^\frac{1}{2^{k+1}} = \left( (a_1 a_2 \ldots a_{2^k} a_{2^k+1})^\frac{1}{2^k} (a_{2^k+2} \ldots a_{2^{k+1}})^{\frac{1}{2^{k+1}}} \right)^\frac{1}{2}\)
\[\left( \frac{a_1 + a_2 + \ldots + a_{2^k} + a_{2^k+1} + a_{2^k+2} + \ldots + a_{2^{k+1}}}{2^{k+1}} \right)\]
using the same trick established right at the outset and the induction hypothesis.

Thus \(P(k+1)\) is true and we have shown by induction that the relationship holds for integers of the form \(n = 2^k\). We need to extend the proof to cover integers other than those of the form \(n = 2^k\). The basic idea is to choose \(n < 2^k\) and somehow extend the sequence of numbers so that you can use the result proved above for integers of the form \(n = 2^k\). In essence we are using backwards induction which involves assuming that if the formula hold for \(n\) it holds for \(n-1\) and that it holds for an infinity of positive integers \(n\), then it is true for any \(n\) (see the main work for the detailed logic).

We want a sequence that looks like this:  \(a_1, a_2, \ldots, a_n, \ldots, a_{2^n}\) because we can then work with that. You may recall that in the dodgy proof (Problem 28) of the AM-GM theorem our intrepid scholar used a trick of padding out a sequence with \(1\)'s. Here is the relevant line:  \((a_{n+1} \cdot \ldots \cdot 1)^\frac{1}{(n+1)-n} = \frac{\text{average of the } a_i\text{ 's}}{n}\) (there being \((n-1)\) \(1\)'s)

Perhaps if we try a similar approach we can get somewhere. Let’s define \(\beta_i = a_i\) for \(1 \leq i \leq n\) where \(n < 2^k\).

For \(n < i \leq 2^k\) define  \(\beta = \frac{a_1 + a_2 + \ldots + a_i}{i}\). In other words  the \(\beta_i\) are simply the average of the \(a_i\). So all we have done is extend the original sequence with copies of the average of the \(a_i\).

At this stage we have a sequence that looks like this:  \(a_1, a_2, \ldots, a_n, \beta, \beta, \ldots, \beta\). How many \(\beta\)'s are there? Simply \(2^k - n\). So we have this to play with:  \((a_1 a_2 \ldots a_n \beta^{2^k-n})^{\frac{1}{2}}\) and using the result already proved above we can say with confidence that
\[\left( a_1 a_2 \ldots a_n \beta^{2^k-n} \right)^\frac{1}{2} \leq \frac{a_1 + a_2 + \ldots + a_n + \beta^{2^k-n}}{2^n} \beta^{\frac{2^k-n}{2^n}} = \frac{\beta^{2^k-n} + \beta}{2^n} \beta^{\frac{2^k-n}{2^n}} = \beta\]

This is looking pretty good. Let’s remove the \(\beta\)'s from both sides and see what we end up with:
\[\left( a_1 a_2 \ldots a_n \beta^{2^k-n} \right)^\frac{1}{2} \leq \frac{\beta^{2^k-n} + \beta}{2^n} \beta^{\frac{2^k-n}{2^n}} \]
\[\left( a_1 a_2 \ldots a_n \right)^\frac{1}{2} \leq \beta^{\frac{2^k-n}{2^n}} \beta^{\frac{2^k-n}{2^n}} \leq \beta\]
\[\left( a_1 a_2 \ldots a_n \right)^\frac{1}{2} \leq \beta\]

So  \((a_1 a_2 \ldots a_n)^\frac{1}{2^n} \leq \beta \beta^{\frac{2^k-n}{2^n}} = \beta^{\frac{2^k}{2^n}} \beta^{\frac{n}{2^n}} = \beta^{\frac{2^k}{2^n}} \beta^{\frac{n}{2^n}} = \beta^{\frac{2^k}{2^n}} \beta^{\frac{n}{2^n}} = \beta^{\frac{2^k-n}{2^n}} \leq \beta\)

Now raise both sides to the power \(\frac{n}{2^k}\) so that we get a statement about the power \(n\) (which is less than \(2^k\) you will recall)

When we do this we get:
\[\left( a_1 a_2 \ldots a_n \right)^n \leq \beta = \frac{a_1 + a_2 + \ldots + a_n}{n}\]
which is what we wanted to prove.

What we did was first prove the AM-GM by induction for any positive integer of the form \(2^k\). We then backward “extended” the proof to numbers not of that form which were less than \(2^k\) but using a padded sequence of numbers \(2^k\) in length so that our inductive proof is still valid based on the concept of backward induction.

30. Arithmetic-Geometric mean in disguise

A 4 Unit HSC problem in 1985 posed the following questions:

(a) Show that for \(k \geq 0\), \(2k+3 > 2\sqrt{(k+1)(k+2)}\)
(b) Hence prove that for \( n = 1, \quad 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2 \left( \sqrt{n + 1} - 1 \right) \)

c) Is it true that for all positive integers \( n, \quad \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 10^{10} \)? Give reasons for your answer.

**Solution**

(a) \((2k + 3)^2 - 4(k + 1)(k + 2) = 4k^2 + 12k + 9 - 4k^2 - 12k - 8 = 1 > 0\)

So \((2k + 3)^2 > 4(k + 1)(k + 2)\)

Now since \( k \geq 0 \), all components of the above inequality are \( > 0 \) so we can take positive square roots to get: \( 2k + 3 > 2\sqrt{(k + 1)(k + 2)} \)

(b) For \( n = 1 \) we have that:

\[
\text{LHS} = 2 \sqrt{1} = 2 \times 1.4142 - 1 < 0.84 < 1
\]

ie LHS > RHS and hence the formula is true for \( n = 1 \)

Assume the formula is true for any \( n \). Then, \( 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > 2 \left( \sqrt{n + 1} - 1 \right) + \frac{1}{\sqrt{n+1}} \) (using the induction hypothesis)

\[
= 2\left(\sqrt{n + 1} - 1\right) + \frac{2n+3-2\sqrt{n+1}}{\sqrt{n+1}}
\]

From (a), \( 2n+3 > 2\sqrt{(n + 1)(n + 2)} \) so \( \frac{2n+3-2\sqrt{n+1}}{\sqrt{n+1}} > \frac{2\sqrt{(n+1)(n+2)} - 2\sqrt{n+1}}{\sqrt{n+1}} = 2\left(\sqrt{n+2} - 1\right) \)

ie \( 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} > 2\left(\sqrt{n+2} - 1\right) \)

So the formula is true for \( n+1 \) and hence true for all \( n \).

(c) We know that \( \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > 2 \left( \sqrt{n + 1} - 1 \right) \)

If there is an \( N \) such that \( 10^{10} = 2 \left( \sqrt{N + 1} - 1 \right) \) then it is false that \( \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 10^{10} \) for all \( n \). By our assumption:

\[
\left( \frac{10^9}{2} + 1 \right)^2 = N + 1
\]

ie \( N = \left( \frac{10^9}{2} + 1 \right)^2 - 1 \). Since \( 2 \) clearly divides \( 10^{10} \) \( N \) is indeed an integer and when it has this value \( \sum_{k=1}^{n} \frac{1}{\sqrt{k}} > 2 \left( \frac{10^9}{2} + 1 \right) > 10^{10} \), hence falsifying the claimed relationship.

So why have I claimed that this is the arithmetic-geometric mean theorem in disguise? Consider the non-negative numbers \( n+1 \) and \( n+2 \) where \( n \geq 1 \).

We know that the arithmetic mean (AM) equals \( \frac{n+1 + n+2}{2} = \frac{2n+3}{2} \). The AM- GM theorem tells us that \( \text{GM} \leq \text{AM} \) ie \( \sqrt{(n+1)(n+2)} \leq \frac{2n+3}{2} \)

ie \( 2n + 3 \geq 2\sqrt{(n+1)(n+2)} \) Whooshka!!

Look familiar? I rest my case. Indeed, teachers who know more than you know can easily crank out a limitless set of problems based on the Cauchy-Schwarz formula and the AM-GM theorem. Now you know the secret. All you have to do is notice the structure. Maths magazines are full of inequality problems based on the AM-GM theorem.

### 31. A Tripos trilogy to send you troppo - three examples from the Cambridge Mathematics Tripos examination

Cambridge has traditionally run a demanding set of exams called the Tripos. In the 19th century the Tripos was notorious for its difficulty and the people who obtained 1st class honours were called Wranglers, some of whom held de facto academic positions. The exams emphasised technique and speed of solution. Some famous people who changed the world were not Wranglers - James Clerk Maxwell (of Maxwell's equations fame) being a prime example. The following three problems come from the Tripos and are in GH
Hardy’s book “A Course of Pure Mathematics” 10th edition, Cambridge University Press, 2006 page 34. If you have understood the arithmetic-geometric mean theorem you should be able to work the first two of them out. The proofs are mine. There are other ways of proving the results but I suspect the proofs I give for the first two are the ones that someone steeped in technique would have used, given that I was “in the AM-GM zone” when I chose them. The third problem can be proved in various ways - I have chosen a combinatorial style of proof which, though more involved than other approaches, provides other insights. Hardy in his book “Inequalities” by G H Hardy, J E Littlewood and G Polya, 2nd edition, Cambridge Mathematical Library, 1952 gives a hint for the proof of the third inequality which involves Burnside's Theorem but that requires too much overhead- see problem 151 page 107).

There is a third approach which is explained in the solution.

(a) **Tripos 1926**: If a and b are positive with $a + b = 1$, then $\left( a + \frac{1}{a} \right)^2 + \left( b + \frac{1}{b} \right)^2 \geq \frac{25}{2}$

(b) **Tripos 1932**: If a, b, c are positive with $a + b + c = 1$, then $\left( \frac{1}{a} - 1 \right) \left( \frac{1}{b} - 1 \right) \left( \frac{1}{c} - 1 \right) \geq 8$

(c) **Tripos 1909**: If $a_i > 0$ for all i and $s_n = a_1 + a_2 + \ldots + a_n$ then $(1 + a_1)(1 + a_2)\ldots(1 + a_n) \geq \sum \frac{s_i^3}{2!} + \ldots + \frac{s_i^n}{n!}$

**Solutions**

(a) **Tripos 1926**: If a and b are positive with $a + b = 1$, then $\left( a + \frac{1}{a} \right)^2 + \left( b + \frac{1}{b} \right)^2 \geq \frac{25}{2}$

Starting with the rather innocuous proposition that $(x - y)^2 \geq 0$ for all x and y, it follows that:

$$\frac{x^2 + y^2}{2} \geq \left( \frac{x + y}{2} \right)^2.$$  

If you doubt this just expand and rearrange. Let $x = a + \frac{1}{a}$ and $y = b + \frac{1}{b}$ then you have:

$$\left( a + \frac{1}{a} \right)^2 + \left( b + \frac{1}{b} \right)^2 \geq \frac{1}{2} \left( 1 + \frac{1}{a} + \frac{1}{b} \right)^2 = \frac{1}{2} \left( 1 + \frac{1}{ab} \right)^2$$

where $a + b = 1$ has been used twice.

Now $ab = a(1 - a) = a - a^2$ and this has a maximum on $(0, 1)$ when $a = \frac{1}{2}$.

This follows from basic quadratic theory or calculus

(and if you’ve got this far you won’t have any trouble with that). The maximum is $\frac{1}{4}$. Thus $ab \leq \frac{1}{4}$ for all $a, b > 0$.

So $\frac{1}{ab} \geq 4$ for all $a, b > 0$

Hence $\left( a + \frac{1}{a} \right)^2 + \left( b + \frac{1}{b} \right)^2 \geq \frac{25}{2}$

(b) **Tripos 1932**: If a, b, c are positive with $a + b + c = 1$, then $\left( \frac{1}{a} - 1 \right) \left( \frac{1}{b} - 1 \right) \left( \frac{1}{c} - 1 \right) \geq 8$

$$\left( \frac{1}{a} - 1 \right) \left( \frac{1}{b} - 1 \right) \left( \frac{1}{c} - 1 \right) = \frac{(1 - ab)(1 - bc)(1 - ca)}{abc} \geq (1 - b - a)(1 - c - a)$$

$$= \frac{ab + bc + ca - abc}{abc}$$

since $a + b + c = 1$

$$= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1$$

Now if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$ the result will follow.

Let $x = \frac{1}{a^2}, y = \frac{1}{b^2}$ and $z = \frac{1}{c^2}$ then by the arithmetic-geometric mean theorem you have:

$$\frac{x + y + z}{3} \geq (xyz)^{\frac{1}{3}}$$

$$\text{i.e. } \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 \left( \frac{1}{abc} \right)^{\frac{1}{3}}$$

Now I claim that $\frac{1}{abc} \geq 27$ for all $a, b, c > 0$ satisfying $a + b + c = 1$

Using the AM – GM theorem again we have:

$$\frac{a + b + c}{3} \geq (abc)^{\frac{1}{3}}$$

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But \( a + b + c = 1 \) so we have \( \frac{1}{3} \geq (abc)^\frac{1}{3} \) ie \( \left( \frac{1}{3} \right)^3 = \frac{1}{27} \geq abc \)

So \( \frac{1}{abc} \geq 27 \) as claimed. Putting it all together we get that \( \left( \frac{1}{a} - 1 \right) \left( \frac{1}{b} - 1 \right) \left( \frac{1}{c} - 1 \right) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \geq 3 \left( \frac{1}{abc} \right)^\frac{1}{3} - 1 \) \( 3 \left( \frac{27}{1} \right)^\frac{1}{3} - 1 = 9 - 1 = 8 \)

**As you can see, the AM-GM is really very powerful.** It takes university level multivariate calculus to prove that \( abc \geq \frac{1}{27} \). For instance you use Lagrange multipliers based on the following function: \( f(x,y,z) = xyz \) and constraint \( g(x,y,z) = x + y + z -1 \). You then need to solve the following simultaneously:

\[
\nabla f = \lambda \nabla g \quad \text{where} \quad \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial z} \end{pmatrix} \lambda.
\]

When you do this you establish that the maximum occurs at \( x = y = z = \frac{1}{3} \) and is indeed \( \frac{1}{27} \).

(c) **Tripos 1909:** If \( a_i > 0 \) for all \( i \) and \( s_n = a_1 + a_2 + \ldots + a_n \) then \( (1 + a_1)(1 + a_2)\ldots(1 + a_n) < 1 + \frac{a_1}{1!} + \frac{a_1^2}{2!} + \ldots + \frac{a_1^n}{n!} \) for \( n > 1 \). First some observations. Because \( 1 + x < e^x \) for all \( x > 0 \) it follows that \( (1 + a_1)(1 + a_2)\ldots(1 + a_n) < e^{a_1 + a_2 + \ldots + a_n} = e^{a_1} = 1 + \frac{a_1}{1!} + \frac{a_1^2}{2!} + \ldots + \frac{a_1^n}{n!} \) for \( n > 1 \). This is based on the fact that \( (1 + a_1 x)(1 + a_2 x) = 1 + (a_1 + a_2) x + a_1 a_2 x^2 \) \( < 1 + s_2 x + \frac{x^2}{2!} \) where \( s_2 = a_1 + a_2 \) since \( a_1 a_2 < \frac{(a_1 + a_2)^2}{2} \) where \( a_1 \) and \( a_2 \) are positive. This is probably the way the problem would have been solved in the heat of an exam.

A perfectly legitimate approach to this type of problem is to play around with low order cases to get a feel for the structure of what is going on. This leads to some further insights of a combinatorial nature which can prove useful in other contexts. Let’s do this for \( n = 3 \).

\[
\begin{align*}
\prod_{i=1}^3 (1 + a_i) &= (1 + a_1)(1 + a_2)(1 + a_3) = 1 + (a_1 + a_2 + a_3) + (a_1 a_2 + a_2 a_3 + a_1 a_3) + a_1 a_2 a_3 \quad \text{[*]} \\
\text{The RHS of [*] can be written using the following notation:} & 1 + \sum_{i=1}^3 a_i + \sum_{i<j} a_i a_j + \sum_{i<j<k} a_i a_j a_k \quad \text{[**]}
\end{align*}
\]

\[
\sum_{i<j} a_i a_j \quad \text{captures the} \quad C_2 \quad \text{distinct dyadic terms.}
\]

The RHS of [**] looks sort of hopeful since \( 1 + (a_1 a_2 + a_2 a_3) + (a_1 a_2 a_3 + a_1 a_3) + a_1 a_2 a_3 = 1 + s_3 + \sum_{i<j} a_i a_j \quad a_i a_j a_k \quad \text{[***]} \quad \text{captures the} \quad C_2 \quad \text{distinct dyadic terms.}
\]

If we could prove that \( \sum_{i<j} a_i a_j < \frac{a_1^3}{3!} \) and \( a_1 a_2 a_3 < \frac{a_1^3}{3!} \) we would have our proof. There are several building blocks we need to develop. Here they are.

(i) \( \prod_{i=1}^n (1 + a_i) = 1 + \sum_{i=1}^n a_i + \sum_{i<j} a_i a_j + \sum_{i<j<k} a_i a_j a_k + \ldots + \sum_{i<j<k<l} a_i a_j a_k a_l \quad \text{[**]} \)

We have already shown that this formula holds for \( n = 3 \) (it is also true for \( n = 1 \) and \( 2 \)). Suppose [***] holds for any \( n \).

Then \( \prod_{i=1}^{n+1} (1 + a_i) = [1 + \sum_{i=1}^n a_i + \sum_{i<j} a_i a_j + \ldots + \sum_{i<j<k<l} a_i a_j a_k a_l + a_{n+1}] (1 + a_{n+1}) \) using the induction hypothesis and the fact that \( \sum_{i<j<k<l} a_i a_j a_k a_l = b_{i,j,k,l} \)

\[
\text{RHS [***]} = 1 + \sum_{i=1}^n a_i + \sum_{i<j} a_i a_j + \ldots + a_{n+1} + a_{n+1} + a_{n+1} + a_{n+1} \quad \text{[****]}
\]

Since the index of \( a_{n+1} \) is greater than all those in each of the sigma sums it follows that:

\[
\text{RHS [****]} = 1 + \sum_{i=1}^{n+1} a_i + \sum_{i<j} a_i a_j + \sum_{i<j<k} a_i a_j a_k + \ldots + a_{n+1} a_{n+2} a_{n+3} a_{n+4}
\]

So [****] is true for all \( n \) by induction.
Where has this got us? We can now compare each sum in \[ \binom{n}{3} \] with the corresponding terms in \( 1 + \frac{a_1}{3} + \frac{a_2^2}{3} + \ldots \). Note that there \( n+1 \) terms in the latter sum. How do you prove that \( \binom{n}{3} \) has the same number of terms? The products in the sums give rise to the following structure (and will equal this if all the \( a_i = 1 \) ) \( C_{10} + C_{11}^2 + C_{12}^3 + \ldots + C_{10}^n \). There are \( n+1 \) such terms - so we identify \( \sum_{i < j} a_i a_j \) (which comprises \( C_2^3 \) distinct terms) with \( \frac{s_1^2}{21} \) and so on.

(ii) A fundamental building block involves the following assertion: \( \sum_{i < j} a_i a_j = \frac{s_1^2}{21} \) for any \( n > 1 \).

This doesn't require anything high powered. 
\[
\frac{s_1^2}{21} = \frac{1}{2} (a_1^2 + a_2^2 + \ldots + a_n^2) = \sum_{i < j} a_i a_j \]
\[
= \sum_{i < j} a_i a_j + 2 \sum_{k < i} a_i a_k \]
\[
= \sum_{i < j} a_i a_j + 2 \sum_{k < i} a_i a_k \] (using the induction hypothesis)
\[
= \sum_{i < j} a_i a_j + 2 \sum_{i < j} a_i a_j \] so the formula is true by induction.

Thus we know that \( \sum_{i < j} a_i a_j \) for any \( n > 1 \) and this does get us somewhere.

(iii) Let's now consider \( \frac{s_1^3}{3!} = \frac{s_1^3}{3} \) where we have used the fact established above. Now we get into some combinatorial arguments.

Remember the light at the end of the tunnel is that we want to prove that \( \frac{s_1^3}{3!} \) and then we want to do the same for the remaining terms. If we can show this we might be able to develop an inductive argument for the remaining terms. Now \( \frac{s_1^3}{3!} \) and then we want to do the same for the remaining terms. If we can show this we might be able to develop an inductive argument for the remaining terms. Now \( \frac{s_1^3}{3!} \) and then we want to do the same for the remaining terms. If we can show this we might be able to develop an inductive argument for the remaining terms.

Let's choose one of the \( C_2^3 \) terms in \( \sum_{i < j} a_i a_j \). For this term how many ways can we choose an element of \( (a_1 + a_2 + \ldots + a_n) \) so that for the remaining triad in the product \( a_i a_j a_k \) we have \( i_1 < i_2 < i_3 \)? Having made the choice for \( a_i a_j \) there can only be \( (n - 2) \) choices of \( a_k \) so that \( a_i a_j a_k \) with \( i_1 < i_2 < i_3 \). Thus there are \( (n - 2) C_2^3 \) such triadic terms in the product which are distinct if \( i_1 < i_2 < i_3 \). Note that this number of terms is greater than \( C_3^4 \) since :
\[
(n - 2) C_2^3 = \frac{n(n-2)(n-3)}{3!} > 0 .
\]
\( \binom{n}{3} - \binom{n-2}{3} > 0 \) for any \( n \geq 2 \). So what we have is this relationship:

Total number of (positive) triadic terms in the product = \( n C_2^3 \). These terms cover \( \sum_{i < j} a_i a_j a_k \) + other positive triadic terms but there is multiple counting in there is a \( k \) such that: product = \( k \sum_{i < j} a_i a_j a_k \) + other positive triadic terms. In essence \( k \) reflects the multiple counting. In fact \( k = \frac{(n-2) C_2^3}{C_2^3} \) = \( \frac{(n-2)(n-3)}{3!} \) = \( 3 \)

So "the product" \( \frac{s_1^3}{3!} = \frac{s_1^3}{3} > \frac{s_1^3}{3} \sum_{i < j} a_i a_j = \sum_{i < j} a_i a_j a_k + \) some positive number.

At last we have it: \( \frac{s_1^3}{3!} \sum_{i < j} a_i a_j = \frac{1}{3} (a_1 + a_2 + a_3 + a_4) \) \[ a(1,2) + a(2,3) + a(3,4) + a(1,4) + a(2,4) + a(1,3) \]

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Show that for all $x > 0$:
\[
\gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt
\]

(i) Show that $\Gamma(1) = 1$ with a rigorous limit argument. I know this is insulting but do it anyway. It is character building.

(ii) Show that $\lim_{\epsilon \to 0} \gamma(\epsilon) = 0$

(iii) Show that $\Gamma(x+1) = x \Gamma(x)$ for all $x > 0$

(iv) Show that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

(v) $\Gamma(x)$ can be defined this way: $\Gamma(x) = \lim_{a \to 0^+} \frac{a^x e^{-a}}{x(x+1)(x+2)\ldots(x+n)}$ for $x > 0$. Use a limiting argument to show that $\Gamma(x+1) = x \Gamma(x)$

Solution

(i) $\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} \, dt = \int_0^\infty e^{-t} \, dt$ Now we all know that the answer is 1. But prove it rigorously.

\[\int_0^\infty e^{-t} \, dt = 1 - e^{-t}\bigg|_0^\infty = 1.\] This means $\lim_{n \to 0^+} n^x e^{-t} \, dt = 1$. Getting into the analysis zone we take $\epsilon > 0$ and demonstrate an $N$ such that for all $n > N$, $\left|\int_n^\infty e^{-t} \, dt\right| < \epsilon$
Now \( \int_0^t e^{-t} \, dt \leq 1 - e^{-t} \leq 1 \) so \( \int_0^t e^{-t} \, dt - 1 \leq e^{-t} - 1 \leq \frac{1}{e^t} \).

Now we want \( \frac{1}{e^t} < \epsilon \) and it is obvious we can choose \( n \) large enough to achieve this but in the interests of accuracy this is how you prove it.

We want \( \frac{1}{e^t} < \epsilon \). Taking logs to the base \( e \):

\[
\log \left( \frac{1}{e^t} \right) < n \log e = n \quad \text{So choose } N > \log e \left( \frac{1}{e^t} \right).
\]

Testing this try \( \epsilon = 10^{-6} \). Thus we need \( n > \log_2 \left( \frac{1}{10^{-6}} \right) = \log_2 10^6 = 13.82 \)

So choose \( n = 14 \) and then \( |e^{-t} - 1| = \frac{1}{e^{14}} + 1 - 1 = \frac{1}{e^{14}} = \frac{1}{102604} = 8.32 \times 10^{-7} < 10^{-6} = \epsilon \)

(ii) Show that \( \lim_{t \to \infty} \frac{t^r}{e^t} = 0 \) where \( t > 0 \) and \( x > 0 \) is fixed

In essence this is saying that \( e^t \) grows much faster than \( t^r \). This is a fundamental result and can be proved several ways. The classic approach is to use calculus and a domination argument. You can actually learn quite a bit by playing with this ratio. For instance, we can rewrite \( \frac{t^r}{e^t} = e^{\ln t - r} \) where \( \ln = \log e \)

Now if you don’t know where the \( e^{\ln t} \) came from here is the quick proof. Let \( v = t^r \) then \( \ln v = x \ln t \) and so going around the circle we get:

\( v = e^{\ln t} \) or \( t^r = e^{\ln t} \)

From \( \frac{t^r}{e^t} = e^{\ln t - r} \) it should be apparent that \( e^{\ln t - r} \to 0 \) as \( t \to \infty \) because \( x \) is fixed and \( \ln t \ll t \) when \( t \) is large (think of the graphs of \( y = \ln t \) and \( y = t \)). The following graph shows the rate of divergence between \( t \) and \( \ln t \) and even when \( \ln t \) is scaled by a fixed \( x \), \( t \) will ultimately dominate. That is the important thing: ultimately \( t \) will dominate \( \ln t \) for fixed \( x \) - see below where \( x = 20 \):

\[
\text{Plot}[[t, 20 \log[t]], \{t, 1, 200\}, \text{PlotStyle} \to \{\}, \text{AbsoluteDashing}[[1.5]], \text{PlotLegend} \to \{t, 20 \log[t]\}, \text{LegendPosition} \to \{1.1, 0\}]
\]

The standard proof of \( \lim_{t \to \infty} \frac{t^r}{e^t} = 0 \) runs this way.

**Part 1.**

\( \lim_{t \to \infty} \frac{\ln x}{t^r} = 0 \) for all \( x > 0 \). We assume \( 0 < s < x \) and \( u \equiv 1 \).

\[
\ln t = \int_1^t \frac{du}{u} \leq \int_1^t u^{s-1} \, du = \left( \frac{u^s}{s} \right)_1^t = \frac{t^s}{s} - \frac{1}{s} \leq \frac{t^s}{s},
\]

Note that since \( u \equiv 1 \) and \( s > 0 \), \( u^s \geq 1 \) and hence \( \frac{1}{s} \leq u^{s-1} = \frac{1}{u^s} \). Incidentally, can you prove rigorously that \( u^s \geq 1 \) if \( u \equiv 1 \) and \( 0 < s < 1 \) in particular? If you did a binomial expansion of \((1 + \epsilon)^x\) where \( \epsilon > 0 \) (ignore the case where \( \epsilon = 0 \)) you would get some negative binomial coefficients. How could you be sure that the overall result is not less than \( 1 \)? On the other hand, if you wrote \( u^s = e^{\ln u} = e^{\alpha s} \) where \( \alpha = \ln u \) you would be pretty confident in asserting that \( u^s \geq 1 \)

So we have \( \frac{\ln t}{t^r} \leq \frac{t^s}{s} \), since \( x > 0 \) (we can divide by \( t^r \) since it is \( > 0 \))

But as \( t \to \infty \), \( \frac{1}{x t^r} \to 0 \) (remember \( x > 0 \)) so we have proved \( \lim_{t \to \infty} \frac{\ln t}{t^r} = 0 \) for all \( x > 0 \)

**Part 2:**

Let \( t = \ln u \) so that as \( t \to \infty \), \( u \to \infty \). Take \( x > 0 \) then \( t^r e^{-t} = \frac{\ln u^r}{u^r} = \left( \frac{\ln u}{u^r} \right)^r \)

But from Part 1 we know that \( \frac{\ln u}{u^r} \to 0 \) since we proved the proposition for any \( x > 0 \) (and \( 1/x \) is \( > 0 \) ). Clearly \( \frac{\ln u}{u^r} \to 0 \) and the limit is established. Note that we have implicitly used the result that if \( \lim_{x \to a} f(x) = A \) then \( \lim_{x \to a} f^n(x) = A^n \) for integral \( n \geq 1 \). Can we prove the limit without recourse to calculus? The answer is "yes". We are concerned with \( \frac{t^r}{e^t} \) when \( t \) is large so let it be of the form \( 2^M \) where \( M \) is large. Then we have something that looks like this:

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\[ \frac{2^M}{e^M} = \left( \frac{2^M}{e^M} \right)^{\frac{1}{2}} < \left( \frac{2^M}{e^M} \right)^{\frac{1}{2}} = 2^{(x-1)M} \frac{1}{2^{(x-1)}} = 2^{M-x-M} = 2^{M-2^M} \]

Now \(2^{M-2^M} \to 0\) since \(M < 2^M\). This is essentially the same observation as was made above in relation to \(x \ln t - t\) when \(t\) is large.

Here is a graph which shows the relationship between \(20M\) and \(2^2M\)

![Graph showing relationship between 20M and 2^2M](image)

(iii) Show that \(\Gamma(x+1) = x \Gamma(x)\) for all \(x > 0\)

Now that we have the building blocks we can do the required integration by parts:

\[ \Gamma(x+1) = \int_0^\infty e^{-t} d(t) = \int_0^\infty e^{-t} t^{x} d(t) = -e^{-t} t^{x} \bigg|_0^\infty + \int_0^\infty t^{x-1} e^{-t} d(t) = 0 + x \int_0^\infty t^{x-1} e^{-t} d(t) = x \Gamma(x) \]

using the limit result from (ii)

(iv) Show that \(\Gamma(n+1) = n! = n(n-1)(n-2)...2.1\) by induction for integral \(n \geq 1\)

From (iii) we know that \(\Gamma(x+1) = x \Gamma(x)\) for any \(x > 0\).

So, in particular, \(\Gamma(n+1) = n \Gamma(n)\) which we assume to be true for any integral \(n \geq 1\). We also know from (i) that \(\Gamma(1) = 1\) so the base case is true.

Hence \(\Gamma(n+2) = (n+1)\Gamma(n+1) = (n+1)n! = (n+1)!\) using the induction hypothesis so the proposition is proved true by the principle of induction.

\[(v) \quad \Gamma(x+1) = \lim_{n \to \infty} \frac{n!}{n^{x+1}} = \lim_{n \to \infty} \frac{n!}{n^{x+1}} \frac{n!}{n^{x+1}} = \Gamma(x) \lim_{n \to \infty} \frac{n!}{n^{x+1}} = \Gamma(x) \lim_{n \to \infty} \frac{x^{x+1}}{x^{x+1}} = x \Gamma(x)\]

(we have implicitly assumed that \(\Gamma(x) = \lim_{n \to \infty} \frac{n!}{n^{x+1}} \) exists.)

If you want to read more about gamma, see Julian Havil, "Gamma: Exploring Euler’s constant", Princeton University Press, 2003.

### 33. A step in a larger proof

In 1697 Johann Bernoulli devised a proof for the following equality: \(\int_0^\infty \frac{1}{x^r} \ln x \ dx = \frac{1}{1} + \frac{1}{2^r} + \frac{1}{3^r} + \cdots\)

Part of the proof involved the following assertion: \(\int_0^\infty \frac{1}{x^r} \ln x \ dx = (-1)^r \frac{r!}{(r+1)^{r+1}}\) for \(r\) integral \(\geq 0\)

(i) Prove \(\lim_{x \to 0^+} x^p \ln x = 0\) for all \(p > 0\)

(ii) Prove by induction \(\int_0^\infty \frac{1}{x^r} \ln x \ dx = (-1)^r \frac{r!}{(r+1)^{r+1}}\) for \(r \geq 0\)

**Solution**

(i) Prove \(\lim_{x \to 0^+} x^p \ln x = 0\) for all \(p > 0\)

Recall that in Problem 33 we proved that \(\lim_{x \to 0^+} \frac{\ln x}{x} = 0\) for all \(x > 0\). To keep the notation the same we can rewrite this as \(\lim_{x \to 0^+} \frac{\ln x}{x^p} = 0\) for all \(p > 0\).

Let \(x = \frac{1}{y}\) so that as \(x \to 0^+\), \(y \to \infty\)

Then \(x^p \ln x = \frac{1}{y^p} \ln \left( \frac{1}{y} \right) = -\frac{\ln y}{y^p}\) and from our earlier result we know the limit of this is 0.

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(ii) Prove by induction \[ \int_0^1 x^r \ln^n x \, dx = (-1)^r \frac{r!}{(r+1)^{n+1}} \] for \( r \geq 0 \).

This integral cries out to be integrated by parts:

\[
\int_0^1 x^r \ln^n x \, dx = \int_0^1 \ln^n x \, d\left(\frac{1}{r+1} x^{r+1}\right) = \left[ \frac{1}{r+1} x^{r+1} \ln^n x \right]_0^1 - \frac{r}{r+1} \int_0^1 \frac{x^{r+1}}{x} \ln^{n-1} x \, dx = -\frac{r}{r+1} \int_0^1 x^r \ln^{n-1} x \, dx
\]

The first part on the right-hand side is zero since by using our result from (i) we know that \( \lim_{x \to 0^+} x^n \ln x = 0 \) so

\[
\lim_{x \to 0^+} x^n \ln^n x = \lim_{x \to 0^+} (x \ln x^n)^r.
\]

This is of the form \( \lim_{x \to 0^+} f(x) g(x) \) where both \( \lim f(x) \) and \( \lim g(x) \) exist and are finite. Thus we can say that \( \lim_{x \to 0^+} (x \ln x^n)^r = \)

\[
\lim_{x \to 0^+} (x \ln x)^r \lim x = \left( \lim_{x \to 0^+} (x \ln x) \right)^r \lim x = 0.0 = 0.
\]

You might have glossed over all of that but those steps technically underpin the disappearance of this term \( \int_0^1 \frac{1}{r+1} x^{r+1} \ln^n x \, dx \) and its cousins in the calculation below. By repeating the integration by parts a second time we get:

\[
\int_0^1 x^r \ln^n x \, dx = -\frac{r}{r+1} \int_0^1 x^r \ln^{n-1} x \, dx
\]

\[
= -\frac{r}{r+1} \left[ \frac{1}{r+1} x^{r+1} \ln^n x \right]_0^1 + \frac{r}{r+1} \int_0^1 \left( \frac{r+1}{r+2} \right) x^{r+1} \ln^{n-2} x \, dx
\]

\[
= \frac{r(r-1)}{(r+1)^2} \int_0^1 x^r \ln^{n-2} x \, dx
\]

\[
= (-1)^r \frac{r!}{(r+1)^{n+1}} \int_0^1 x^r \, dx \quad \text{At least this is the guess because after each iteration the index of } \ln^n x \text{ drops by 1 and so after } r \text{ iterations it will be zero.}
\]

\[
= (-1)^r \frac{r!}{(r+1)^{n+1}} \quad \text{after doing the last integration (i.e. } \int_0^1 x^r \, dx = \frac{1}{r+1} \text{)}
\]

To see this in a more detailed (laborious?) fashion let \( J(n,m) = \int_0^1 x^n \ln^m x \, dx \)

Then \( J(n, m) = \int_0^1 \ln^m x \, d\left(\frac{x^n}{n+1}\right) \)

\[
= \frac{x^n}{n+1} \ln^m x \bigg|_0^1 - \int_0^1 \frac{x^n}{n+1} \ln^{m-1} x \, dx
\]

\[
= -\frac{m}{n+1} \int_0^1 x^n \ln^{m-1} x \, dx
\]

\[
= -\frac{m}{n+1} J(n, m-1)
\]

Let's do it again so that \( J(n, m-1) = \int_0^1 x^n \ln^{m-1} x \, dx \)

\[
= \int_0^1 \ln^{m-1} x \, d\left(\frac{x^n}{n+1}\right) \]

\[
= -\frac{m-1}{n+1} \int_0^1 x^n \ln^{m-2} x \, dx \quad \text{where we repeatedly use the fact that } x^{n+1} \ln^{m-1} x \to 0 \text{ as } x \to 0^+
\]

\[
= -\frac{m-1}{n+1} J(n, m-2)
\]

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\[ J(n,m) = \frac{m}{n+1} J(n,m-1) = \frac{m(m-1)}{(n+1)^2} J(n,m-2) \]

This structure suggests that after the kth iteration we would have:

\[ J(n,m) = (-1)^k \frac{m(m-1) \ldots (m-k+1)}{(n+1)^k} J(n,m-k) \]

If we let \( k = m \) we get \( J(n,m) = (-1)^m \frac{m!}{(n+1)^m} J(n,0) \)

\[ = (-1)^m \frac{m!}{(n+1)^m} \int_0^1 x^n \ln^m x \; dx \]

\[ = (-1)^m \frac{m!}{(n+1)^m} \frac{1}{n+1} \]

\[ = (-1)^m \frac{m!}{(n+1)^{m+1}} \]

This is an example where induction gives us some confidence our guess is correct. There is no necessary connection between \( n \) and \( m \). Let us fix \( m \) as an arbitrary integer \( 0 \) and perform an induction on \( m \). We are trying to show that \( J(n,m) = \frac{m!}{(n+1)^m} J(n,m-1) \) holds for all \( m \) assuming \( n \) is fixed but arbitrary. Now the base case is where \( m = 0 \) i.e. \( J(n,0) = \int_0^1 x^n dx = \frac{1}{n+1} = (-1)^0 \frac{0!}{(n+1)^0} \).

As the induction hypothesis assume that \( J(n,m) \) is true for all \( m \) (assuming a fixed but arbitrary \( n \)) i.e. \( J(n,m) = (-1)^m \frac{m!}{(n+1)^m} \). In effect this is saying \( J(1,m) \) is true for all \( m \), \( J(2,m) \) is true for all \( m \) etc. Then \( J(n,m+1) = \frac{1}{n+1} \int_0^1 x^n \ln^{m+1} x \; dx \). In this integral let \( du = x^n \; dx \) so that \( u = \frac{x^{n+1}}{n+1} \) and \( v = \ln^{m+1} x \) so that \( dv = (m+1) \ln^m x \frac{dx}{x} \). Thus:

\[ J(n,m+1) = -\frac{(m+1)}{(n+1)} \int_0^1 x^n \ln^m x \; dx \]

\[ = (-1)^m \frac{(m+1)!}{(n+1)^{m+1}} \]

Thus we know that we can choose an arbitrary integer \( n \) \( 0 \) and for any such \( n \), \( J(n,m) = (-1)^m \frac{m!}{(n+1)^m} \) will hold for all \( m \) \( 0 \). So let \( n = m = r \) and so \( J(r,r) = (-1)^r \frac{r!}{(r+1)^r} \).

If you start with \( J(r+1, r+1) \) and try to use an induction hypothesis of the form \( J(r, r) \) you will find that you get \( \int_0^1 x^{r+1} \ln^{r+1} x \; dx = \frac{1}{r+1} J(r+1, r) \). But the index for \( x \) is too high and we know from what has been done above that it won’t go down. So the induction hypothesis doesn’t actually get used. However, by fixing \( n \) independently of \( m \) the result follows.

34. Bernoulli’s inequality

Suppose \( x \) is a real number \( -1 \), then \((1 + x)^n = 1 + nx \; \forall n \in \mathbb{N} \)

**Solution**

The statement is trivially true for \( n = 0 \) and only slightly more interestingly true for \( n = 1 \). The more interesting base case is where \( n = 2 \). In that case the LHS \( = 1 + 2x + x^2 \geq 1 + 2x \) since \( x^2 \geq 0 \) for \( x \geq -1 \).

Assume the formula is true for any \( n \), then:
\[(1 + x)^{n+1} = (1 + x)(1 + x)^n \geq 0\]

\[(1 + x)(1 + nx) \text{ using the induction hypothesis and the fact that } 1 + x \geq 0 \text{ if that were not the case what would happen?} \]

\[= 1 + nx + x + nx^2 \]

\[= 1 + (n + 1)x + nx^2 \]

\[\geq 1 + (n + 1)x \text{ since } nx^2 \geq 0 \forall x \geq -1 \]

Accordingly the formula is true for all \(n\).

### 35. A generalisation of Bernoulli’s inequality

If \(\alpha, \beta, \gamma, \ldots \lambda\) are greater than -1, and are all positive or all negative, then \((1 + \alpha)(1 + \beta)(1 + \gamma)\ldots(1 + \lambda) > 1 + \alpha + \beta + \gamma + \ldots + \lambda\)

Clearly if \(\alpha = \beta = \gamma = \ldots = \lambda\) then you have Bernoulli’s inequality.

Prove the inequality (this problem comes from G Hardy, J E Littlewood and G Polya, “Inequalities” by Hardy, Littlewood, Cambridge University Press, 2001, page 60)

**Solution**

The first point to note that the problem necessarily involves at least 2 terms since it is false that \(1 + \alpha > 1 + \alpha\). For convenience let’s work with \(x_1, x_2, \ldots, x_n\).

The formula is true for \(n = 2\) as the base case since \((1 + x_1)(1 + x_2) = 1 + x_1 + x_2 + x_1x_2 > 1 + x_1 + x_2\) where we have used the fact that \(1 + x_1 > 0\) and \(1 + x_2 > 0\) and the fact that both are either positive or negative to ensure that the product \(x_1x_2\) is positive. Now consider the product of \(n+1\) terms:

\[(1 + x_1)(1 + x_2)\ldots(1 + x_n)(1 + x_{n+1}) > (1 + x_1 + x_2 + \ldots + x_n + x_{n+1}) = 1 + x_1 + x_2 + \ldots + x_n + x_{n+1} + x_1x_{n+1} + x_2x_{n+1} + \ldots + x_nx_{n+1}\]

\[> 1 + x_1 + x_2 + \ldots + x_n + x_{n+1}\]

using the same reasoning as given in the base case.

Accordingly the formula is true for all \(n\).

### 36. Subsets

Let \(S(n)\) be the number of subsets of an \(n\)-set \(\{1, 2, 3, \ldots, n\}\). Consider \(S(n+1)\) which we calculate by noting that taking all the subsets of \(\{1, 2, 3, \ldots, n\}\) and extending each by either doing nothing or adding the element \(n+1\). Hence \(S(n+1) = S(n)\) since for each of the elements of \(\{1, 2, 3, \ldots, n\}\) there are 2 ways of performing the extension.

Using the recurrence relation \(S(n+1) = 2S(n)\) make an inspired guess at the size of the power set of \(\{1, 2, 3, \ldots, n, n+1\}\) (ie the set of all of its subsets) and prove your guess by induction.

Which is greater: the number of permutations of a set of \(n\) elements or the number of subsets? Prove your conjecture by induction.

**Solution**

\(S(n+1) = 2S(n) = 2^2S(n-1) = 2^3S(n-2) = \ldots = 2^nS(1) = 2^n2 = 2^{n+1}\) since there are 2 subsets of \(\{1\}\), namely itself and the empty set. This suggests our guess for \(S(n)\) should be \(2^n\).

Clearly \(S(1) = 2\) as already noted. Suppose \(S(n) = 2^n\) for any \(n \geq 1\).

Then \(S(n+1) = 2S(n) = 2^{n+1}\) using the induction hypothesis and the result follows.

The second part of the problem asks whether \(n! > 2^n\). First try some low numbers and you will see that for \(n = 1, 2\) and 3 the number of permutations is less than the number of subsets but \(n! > 2^n\) for \(n = 4, 5, \ldots\). Thus the conjecture is that \(n! > 2^n\) for \(n \geq 4\) (our induction hypothesis).

Our base case is thus \(n = 4\). Now \((n+1)! = (n+1)n! > (n+1)2^n > 2^{n+1}\) using the induction hypothesis and the fact that \(n+1 > 2\). Thus the conjecture is true for \(n+1\).

Hence the conjecture is established as true for all \(n \geq 4\).

### 37. Recurrence relations and probability theory

In probability theory it is very common to solve problems using hopefully linear recurrence relations with associated boundary conditions in the probabilities. Students who do actuarial studies and finance theory will be regularly exposed to such techniques. A typical sort of problem which involves setting up a recurrence relation usually runs along the following lines. Such problems have figured in the top HSC maths exams and are usually poorly done judging by the examiners’ comments.
(i) A fair coin is to be tossed repeatedly. For integers \( r \) and \( s \), not both zero, let \( P(r, s) \) be the probability that a total of \( r \) heads are tossed before a total of \( s \) tails are tossed so that \( P(0, 1) = 1 \) and \( P(1, 0) = 0 \). You are asked to explain why for \( r, s \geq 1 \) the following relation holds:

\[
P(r, s) = \frac{1}{2} P(r-1, s) + \frac{1}{2} P(r, s-1)
\]

(ii) Find \( P(2,3) \)

(iii) Prove using induction on \( n = r + s - 1 \) that:

\[
P(r, s) = \frac{1}{2^n} \left( C^n_0 + C^n_1 + \ldots + C^n_{s-1} \right) \text{ for } s \geq 1
\]

**Solution**

It is important to get the boundary probabilities right. Why? Because you will eventually reduce something relatively complicated to something simpler involving a boundary probability (roughly speaking). In effect you “descend” from the top to the boundary probability.

Now (and this is important for this problem) can you assert that \( P(r,0) = 0 \) and \( P(0,s) = 1 \)? In general \( P(r,n) = 0 \) if \( r < n \) and \( P(n,s) = 1 \) if \( s < n \). If you read the historical material below concerning the Problem of Points, Fermat and Pascal approached this problem by considering an extended game in which A needed \( r \) points to win and B needed \( s \) points to win so that to get a winner they will have to continue to play for \( r + s - 1 = n \) games. Thus if \( r < n \), \( P(r,n) \) must be zero and if \( s < n \), \( P(n,s) \) must be 1.

These types of problems involve a standard type of solution which, if followed, can be replicated in more complicated examples than the simple situation referred to above.

In essence you break down the problem into mutually exclusive options or “pathways”. The is the big idea because it allows you to sum probabilities so you get a recurrence equation relationship which can be solved by various techniques.

Thus you ask yourself in what situations will \( r \) heads appear before \( s \) tails.

If your first toss is a head (with probability \( \frac{1}{2} \)) you will need \( (r-1) \) heads before \( s \) tails appear with probability \( P(r-1, s) \). Note that this means that if you have already tossed a head, \( P(1,0) = 0 \) since you would need 0 heads before 0 tails appear and the probability of that is zero. Independence is assumed so this particular pathway has probability \( \frac{1}{2} P(r-1, s) \). Alternatively, if your first toss is a tail with probability \( \frac{1}{2} \), you will need \( r \) heads and \( (s-1) \) tails with probability \( P(r, s-1) \). The compound probability is thus \( \frac{1}{2} P(r, s-1) \).

Now these two pathways exhaust the universe of options and are mutually exclusive so the required probability is:

\[
P(r, s) = \frac{1}{2} P(r-1, s) + \frac{1}{2} P(r, s-1)
\]

(ii) What is \( P(2,3) \) for instance? Just keep applying the recurrence relation until you get to sub-elements that you know. This is tedious and error prone so it is best to do it in chunks:

\[
P(2,3) = \frac{1}{2} \left( P(1,3) + P(2,2) \right)
\]

\[
\frac{1}{2} P(1,3) = \frac{1}{4} P(0,3) + \frac{1}{4} P(1,2)
\]

\[
= \frac{1}{4} \cdot 1 + \frac{1}{8} P(0,2) + \frac{1}{8} P(1,1)
\]

\[
= \frac{1}{4} + \frac{1}{8} \cdot 1 + \frac{1}{16} P(0,1) + \frac{1}{16} P(1,0)
\]

\[
= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \cdot 0
\]

\[
= \frac{7}{16}
\]

\[
\frac{1}{2} P(2,2) = \frac{1}{4} P(1,2) + \frac{1}{4} P(2,1)
\]

\[
= \frac{1}{8} + \frac{1}{16} + \frac{1}{8} P(1,1) + \frac{1}{8} P(2,0) \text{ using the result for } \frac{1}{4} P(1,2) \text{ already derived above}
\]

\[
= \frac{3}{16} + \frac{1}{16} P(0,1) + \frac{1}{16} P(1,0) + \frac{1}{8} \cdot 0
\]

\[
= \frac{3}{16} + \frac{1}{16} \cdot 1 + \frac{1}{16} \cdot 0
\]

\[
= \frac{4}{16}
\]

\[
P(2,3) = \frac{7}{16} + \frac{4}{16}
\]

\[
= \frac{11}{16}
\]
It is important to note that having chosen \( n = r + 1 \), \( s \) must be at least 2 hence the final term in the second curly brackets \( m \):

\[
P(r+1, s) = \frac{1}{2^n} \{ C_0^n + C_1^n + \ldots + C_{s-1}^n \} \quad \text{for } s \geq 1
\]

Now let’s assume the new formula is for any \( n=r \). We have to get the right structure for \( P(r+1, s) \). We know from our basic relation that:

\[
P(r+1,s) = \frac{1}{2} P(r, s) + \frac{1}{2} P(r + 1, s - 1)
\]

We know that \( P(2,3) = \frac{1}{2^2} \{ C_0^2 + C_1^2 + C_2^2 \} \) using the induction hypothesis.

This is where Pascal’s identity comes in: it says, for instance, that \( C_n^{n+1} = C_n^1 + C_n^n \).

Let’s expand \( \frac{1}{2} P(r, s) + \frac{1}{2} P(r + 1, s - 1) \) and see what emerges:

\[
\frac{1}{2} P(r, s) + \frac{1}{2} P(r + 1, s - 1) = \frac{1}{2^{r+1}} \{ C_0^r + C_1^r + C_2^r + \ldots + C_{s-2}^r \} + \frac{1}{2^{r+1}} \{ C_0^{r+1} + C_1^{r+1} + C_2^{r+1} + \ldots + C_{s-2}^{r+1} \}
\]

It is important to note that having chosen \( n = r + 1 \), \( s \) must be at least 2 hence the final term in the second curly brackets must be \( C_{n-2}^n \) (it was at least 1 when \( n = r \)). Now pair up the terms in the first curly brackets with those in the second as follows using Pascal’s Identity:

\[
\begin{align*}
C_{n-1}^r + C_{n-2}^r &= C_{n-1}^{r+1} \\
C_{n-2}^r + C_{n-3}^r &= C_{n-2}^{r+1} \\
C_{n-3}^r + C_{n-4}^r &= C_{n-3}^{r+1} \\
& \vdots \\
C_0^r + C_1^r &= C_0^{r+1}
\end{align*}
\]

So we have:

\[
P(r+1,s) = \frac{1}{2^{r+1}} \{ C_0^{n+1} + C_1^{n+1} + C_2^{n+2} + \ldots + C_{s-2}^{n+1} \} \quad \text{and this has the appropriate structure so the formula is indeed true by induction.}
\]

Note that \( P(2,3) = \frac{1}{2^2} \{ C_0^1 + C_1^1 + C_2^1 \} = \frac{1 + 4 + 6}{16} = \frac{11}{16} \) as before. You could symmetrically do the induction on \( s \).

### 38. Some combinatorial manipulations

Following on from Problem 37 one can go a bit further in the spirit of combinatorial manipulation and assert the following:

\[
\sum_{k=r-1}^{n-s} \frac{1}{2^r} C_k^{r+s} = \frac{1}{2^n} \{ C_0^n + C_1^n + \ldots + C_{s-1}^n \} \quad \text{where } n = r + s - 1 \quad \text{and } s \geq 1
\]

Prove this without induction.

**Solution**

The interesting thing is that this problem is essentially the so-called *Problem of Points* that was discussed in correspondence between Pierre de Fermat and Blaise Pascal in the 17th century. There is a discussion of this issue in an extremely good probability textbook that is available for free on the internet. It is produced by Charles Grinstead and J Laurie Snell and represents what I think is a good approach to the teaching of probability theory. The authors have put a huge amount of effort into developing an interesting and rigorous approach.
Pascal proved the following formula essentially by backwards induction:

$$\sum_{k=r+1}^{n} \frac{1}{2} C_{k}^{n} = \frac{1}{2^n} \{ C_{0}^{n} + C_{1}^{n} + \ldots + C_{r+s-1}^{n} \} \quad \text{where} \ n = r + s - 1 \text{ and } s \geq 1$$

Let n = r + s - 1

Then $$\sum_{k=r+1}^{n} \frac{1}{2} C_{k}^{n} = \frac{1}{2^n} \sum_{k=r+1}^{n} C_{k}^{n}$$

We want to show that $$\sum_{k=r+1}^{n} C_{k}^{n} = \sum_{k=0}^{n} \frac{1}{2} C_{k}^{n}$$

The first point to note that both sums have n - r + 1 s terms. By appropriate pairing of terms the result falls out. For instance, $$C_{s-1}^{n} = \frac{n!}{(n-r+1)!} = \frac{n!}{r!(n-r)!}$$ using the fact that n = r + s - 1

More generally, $$C_{s-1-j}^{n} = C_{r+j}^{n} \text{ for } 0 \leq j \leq s - 1 \text{ and this exhausts the pairs.}$$ The equality is seen as follows:

$$C_{r+1-j}^{n} = \frac{n!}{(n-s+1+j)!((s-1-j)!}$$

$$= \frac{n!}{(r+j)!((n-r-j)!}$$

$$= C_{r+j}^{n} \text{ for all } 0 \leq j \leq s - 1$$

Thus the sums are indeed identical. We got P(2,3) = $\frac{11}{16}$ before using $$\sum_{k=0}^{n} \frac{1}{2} C_{k}^{n}$$. Using the new formula we get:

$$P(2,3) = \sum_{k=2}^{4} \frac{1}{2^{k}} C_{k}^{4} = \frac{6+4+1}{16} = \frac{11}{16}$$

as before.

The problem arose in this context. Two players R and S are playing a sequence of games and the first player to win n games wins the match. What is the probability that R wins the match at a time when R has won r games and S has won s games. Pascal and Fermat considered the problem of determining the fair division of stakes if the game is called off when the first player has won r games and the second player has won s games with r < n and s < n. In this extended game where R needs r games to win and S needs s games to win, the players must play r + s - 1 games to establish a winner and the winner of this extended game would be the winner of the original game. The formula they got was P(r,s) = p P(r+1,s) + q P(r,s+1) where p + q = 1 and P(r,0) = 0 if r < n and P(n,s) = 1 if s < n.

Can you see why r + s - 1 games are needed? If R wins r games then S has won at most s - 1 games and cannot reach is target of s games. However if S wins s games this leaves r - 1 games for R and he can’t meet his target.

Once you pose the problem in the form of the extended game it is clear that each of the relevant paths has probability $$\frac{1}{2^{r+s-1}}$$ and there are $$C_{k}^{r+s-1}$$ ways choosing the k games from r + s - 1 = n so that the required probability is as claimed.

An important generalisation of the approach used in Problem 37

The step in the proof whereby we get the probabilities of the separate pathways actually reflects an important combinatorial technique which was obscured by the fact that we were dealing with probabilities. In what we did above the favourable cases are normalised by the total number of pathways in order to get a probability. Let’s take a situation where we need to work out the number of ways of getting from A to B on a grid. The problem is posed this way. Imagine a grid of u + 1 parallel lines running north-south and v + 1 parallel lines running east-west. If you start at the southwest corner, how many ways are there of reaching the northwest corner by foot or car assuming you can only travel north or east?

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Take an arbitrary intermediate point \((x,y)\). There are only 2 ways of getting to \((x,y)\) given the constraints of the problems (ie northerly and easterly movement):

(a) from \((x-1, y)\); or

(b) from \((x, y-1)\)

Thus if \(N(x,y)\) is the number of ways of getting to \((x,y)\) we should have: \(N(x,y) = N(x-1, y) + N(x, y-1)\) which has the same structure as the recurrence relation arrived at above. The boundary conditions are: \(N(0,1) = N(1,0) = 1, N(0,2) = N(2,0) = 1\) and \(N(1,1) = 2\).

I now make the bold claim that \(N(x,y) = \frac{(x+y)!}{x!y!}\)

Prove this two ways: by induction and by combinatorial reasoning.

**Solution**

Let's prove this two ways. First by induction and then another way which is the basis of a powerful combinatorial technique. First the inductive proof.

The claim is true for \(x + y = 1\) since \(N(0,1) = 1 = \frac{(0+1)!}{0!1!} = N(1,0)\)

Now suppose the claim is true for any positive integral \(k = x + y\) i.e \(N(x,y) = \frac{(x+y)!}{x!y!}\)

Now when \(x + y = k + 1\) we will have:

\[
N(x, y) = N(x, y - 1) + N(x - 1, y)
\]

\[
= \frac{k!}{x!(k-x)!} + \frac{k!}{(x-1)!(k-1-x)!} \quad \text{(using the induction hypothesis)}
\]

\[
= \frac{k!}{x!(k-x)!} + \frac{k!}{(x-1)!(k+1-x)!} \quad \text{(using the fact that } x + y = k + 1\text{)}
\]

\[
= \frac{k!}{x!(k-x)!} \left( 1 + \frac{x}{k+1-x} \right)
\]

\[
= \frac{k!(k+1)}{x!(k-x)!(k+1-x)}
\]
\[
\frac{(k+1)!}{x!(k+1-x)!} = \frac{k!}{x!(k+1-x)!} + \frac{1}{x!(k+1-x)!}
\]

So the formula is true for \(k+1\) and hence is true for all \(k\) by the principle of induction.

Now for the second proof which illustrates an important technique. Suppose we start at \((0,0)\) and want to get to \((x,y)\). It will take exactly \(x+y\) moves to do so. Of that total number of moves \(x\) have to be in an easterly direction and \(y\) have to be in a northerly direction.

What is going on here is that we are asserting a bijection between the set of paths and the set of distinct arrangements of \(x\) “norths” and \(y\) “easts”, eg NNEENENNE. Hence it must be the case that \(N(x,y) = C^x+y_x = C^x+y_y\).

A bijective function places two sets into a one to one correspondence. If \(f : A \rightarrow B\) is such that \(f(A) = B\) then \(f\) is surjective or onto (ie every element of \(B\) gets “hit”). If for every two distinct elements \(a_1\) and \(a_2\) of \(A\) their images are distinct (ie \(a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)\)) we say that \(f\) is injective or one-to-one. A bijective function is surjective and injective. If there is a bijection between two finite sets they must have the same number of elements.

39. Proving by induction that no-one ever escapes from the gulag of chartered accounting

This is a problem which involves using induction to prove that no-one ever escapes from a certain chartered accounting firm that has a certain set of rules. Here is the problem.

Each new aspiring chartered accountant (CA) is given a bowl of 15 red beans and 12 green beans (to count of course). The head audit partner (nicknamed “Gulag” by his employees) demands at regular intervals that each CA produce a time sheet of chargeable units (a chargeable unit is 6 minutes - this has nothing to do with the problem - it is just letting you know what you’re in for if you become a CA). Gulag bellows out “Time sheets” and the cowering CAs dutifully produce them. Like the crazed captain of a ship cast adrift on the endless seas of chartered accountancy, Gulag offers the young CAs hope of escaping from the mind numbing tedium of tax schemes, sets of accounts and journal entries. He has two rules one of which must be done when he bellows “Timesheets”:

**Rule 1:** If a CA holds at least 3 red beans, the CA may remove 3 red beans and add 2 green beans.

**Rule 2:** The CA may replace each green bean with a red bean and replace each red bean with a green bean. That is, if the CA starts with \(i\) red beans and \(j\) green beans, then after Rule 2 is performed the CA will have \(j\) red beans and \(i\) green beans.

The head of audit offers the CAs this deal: they can leave the firm only when they have exactly 5 red beans and 5 green beans.

One of the CAs has a friend who has studied mathematics and he asks the friend whether these two rules will enable anyone to escape from chartered accounting. The maths friend thought about the problem for a while and came up with this:

**Theorem:** Given the head of auditor’s rules, no-one ever leaves chartered accountancy

Prove this theorem using induction.

**HINT:** You need to search for some quantity that arises from the various pathways that is invariant under each operation of the rules. You then have to show that this invariant condition is violated by the head of audit’s condition for escape.

* GULAG was the Russian agency that operated the forced labour camps under Stalin. They were not enjoyable places - neither are chartered accounting firms.

**Solution**

There is a starting point with 15 red beans and 12 green beans which I will write as \((15,12)\). In general a subsequent position will be \((r,g)\). The hint involves having a look at how the pathways develop based on which of rules 1 or 2 are invoked each time Gulag bellows “Timesheets”. Here are some of the pathways:
There are really only two possible candidates for some invariant function of the pathways and they are $\Delta = r - g$ and $\Sigma = r + g$. Now when you look at how $\Sigma$ varies with each step in the pathways it does not give rise to the sort of invariance we want. What we want to be able to do is to say that after the head of audit bellows "Timesheets" $n$ times the invariant quantity has a specific form which must hold at the start. If we focus on the top pathway after five iterations you will have the sequence $(15,12) \rightarrow (12,14) \rightarrow (9,16) \rightarrow (6,18)$ etc. The corresponding $\Sigma$ sequence is $27 \rightarrow 26 \rightarrow 25 \rightarrow 24$ and in principle we want to be able to say that the function on that pathway will give us the correct form every iteration. $\Sigma$ can’t do this because it has to be even at times and odd right at the beginning (as well as at other times). So let’s run with $\Delta = r - g$. When one looks at the values of $\Delta$ it is clear that they are of two forms: $5k + 2$ or $5k + 3$ for some integer $k$ (which could be negative). So let’s run with $\Delta = r - g$.

Our inductive proposition $P(n)$ is that after $n$ bellows by the head of audit $\Delta$ is either of the form $5k + 2$ or $5k + 3$ for some $k \in \mathbb{Z}$.

Our base case $P(0)$ is true since $r - g = 15 - 12 = 5 \times 0 + 3$

The inductive step runs this way. We assume that $P(n)$ holds after $n$ stentorian bellows by the head of audit where $n \geq 0$. As above we assume that $\Delta$ is of the form $5k + 2$ or $5k + 3$ for some integer $k$. After $n + 1$ bellows by the head of audit a CA has two options based on the two rules:

1. If $r = 3$ there may have been a removal of 3 red beans and the addition of 2 green beans so $\Delta’ = (r - 3) - (g + 2) = (r - g) - 5$
   But by assumption $\Delta = r - g = 5k + 2$ or $5k + 3$ so $\Delta’ = 5k + 2 - 5 = 5(k - 1) + 2$. OR $\Delta’ = 5k + 3 - 5 = 5(k - 1) + 3$. So in either case the difference is of the required form. Hence $P(n + 1)$ is true.

2. If the CA does the switch of red and green beans the number of red beans less green beans is $\Delta’ = g - r$. Hence if $\Delta = r - g = 5k + 2$ then:
   $\Delta’ = -5k - 2 = 5(-k - 1) + 3$ and if $\Delta = r - g = 5k + 3$ then $\Delta’ = -5k - 3 = 5(-k - 1) + 2$.
   Thus in either case $P(n + 1)$ is true since the differences have the correct form.

Accordingly, $P(n)$ is true for all $n \geq 0$. What this shows is that the number of red beans minus green beans is always of the form $5k + 2$ or $5k + 3$ but to get out of chartered accounting you need to get 5 red beans and 5 green beans (difference 0) but neither $5k + 2$ nor $5k + 3$ will give this result since $k$ is an integer.

This proves that a CA can’t escape from chartered accounting under the head of audit’s rules.

### 40. Integral equations and inductive proof

As noted in the discussion about Feynman’s path integral approach in the main work, integral equations play an important role in mathematical physics. The area of functional analysis deals systematically with the properties of integral equations. Here is an abbreviated background to the basic theory which focuses on the concept of a contraction mapping since that is the “entry point” level for the theoretical development. A good book on the subject that is both rigorous, well written and with lots of examples is Graeme Cohen, “A Course in Modern Analysis and its Applications”, Cambridge University Press, 2003.

Let $f$ be a function from $[a, b]$ to a subset of $[a, b]$. Suppose that there is a constant $0 < k < 1$ such that for any two points $x_1$, $x_2$ in $[a, b]$
The following holds:

\[ |f(x_1) - f(x_2)| < k \cdot |x_1 - x_2| \]

The function \( f \) is then said to satisfy a Lipschitz condition. The **Fixed Point Theorem** guarantees that the equation \( f(x) = x \) has a unique solution for \( x \) in \([a, b]\). The Fixed Point Theorem states that every contraction mapping on a **complete** metric space has a unique fixed point. The \( k \) being less than 1 gives rise to the contraction concept because when you iterate the process \( n \) times you get a factor \( k^n \) which is small - hence the contraction. The example below shows this iterative process in action.

This is not the place to develop the theory of complete metric spaces (in fact this is the incomplete treatment of complete metric spaces!) , suffice it to say that the concept of a metric involves the idea of a distance in its most general context. Thus if one is interested in studying differentiable functions a measure of "distance" of interest may be

\[ d(f, g) = \max_{x \in [a,b]} |f(t) - g(t)| \]

where \( f \) and \( g \) are differentiable functions on \([a, b] \). In the case of real numbers it could just simply be

\[ d(x_1, x_2) = |x_1 - x_2| \]

The concept of a complete metric space involves the idea that every Cauchy sequence in that space converges. A sequence is a Cauchy sequence if for every \( \epsilon > 0 \) there exists a positive integer \( N \) such that \( d(x_n, x_m) < \epsilon \) whenever \( m, n > N \). Here \( d(x_n, x_m) \) is some way of describing a consistent distance measure (metric) on the space. Note that the space can be inhabited by functions, sequences etc rather than just plain old numbers.

The contraction mapping principle has many applications. For instance it underpins the proof of the existence of solutions to the first order differential equation \( \frac{dy}{dx} = f(x, y) \) with initial conditions \( y = y_0 \) and \( x = x_0 \). 

**Fixed point theorems** are interesting in themselves. John Nash of “Beautiful Mind” fame used a fixed point theorem to establish the existence of a social equilibrium, that is, a situation in which no household can individually improve its well-being by altering the amount it consumes of various items. Nash shared the Nobel Prize for Economics in 1994 for his mathematical work in this area.

We know from the Intermediate Value Theorem that if a function \( g \) is continuous on \([0, 1]\) such that \( g(0) = 0 \) and \( g(1) = 0 \) then there must be some \( x \) in \([0, 1]\) such that \( g(x) = 0 \). Now suppose that \( f : [0, 1] \to [0, 1] \) is continuous. Let \( g(x) = f(x) - x \) so that \( g(0) = 0 \) and \( g(1) = 0 \). Since \( g \) is also continuous there must be some \( x \) such that \( g(x) = 0 \) ie \( f(x) = x \). This \( x \) is a fixed point of \( f \). Note that continuity was essential in establishing the existence of the fixed point. This principle is generalized to more abstract environments. For instance **Brouwer's Fixed Point Theorem** ("Brouwer" is usually pronounced "Brow-ver") states that if \( f: B^n \to B^n \) (where \( B^n \) is the unit ball in \( \mathbb{R}^n \)) is continuous it must have a fixed point. This theorem underpins the following observation. If a circular disk made of rubber were placed on a table and then you took the sheet and deformed it by folding and stretching and put it back within the circle where it started, there is always a point that is fixed ie it ends up in the same place. The trouble is that while you may know that such a point exists you may not be able to specifically identify it.

**Problem**

It can be shown for instance that an equation of the form \( x(s) = \frac{1}{2} \int_0^s t x(t) \, dt + \frac{5s}{6} \) has a unique solution for \( s \) \in \([0, 1]\). Take this on faith and assume that the starting point for the iterative process is the function \( x_0(s) = 1 \) for \( 0 < s < 1 \). Define \( x_n(s) = \frac{1}{2} \int_0^s t x_{n-1}(t) \, dt + \frac{5s}{6} \)

I’ll get you going with \( x_1(s) \).

\[ x_1(s) = \frac{1}{2} \int_0^s t x(t) \, dt + \frac{5s}{6} = \frac{1}{2} \cdot \frac{1}{2} + \frac{5s}{6} = \frac{26s}{24} = \frac{13s}{12} \]

(i) Now find \( x_2(s) \) and \( x_3(s) \) and then **guess** a form for \( x_n(s) \) and then prove your guess by induction.
(iii) Suppose \( f : \mathbb{R} \to \mathbb{R} \) has the property that there is a positive constant \( k \) such that \( |f(u) - f(v)| \leq k (u - v)^2 \). Prove that \( f \) is a constant function. Note that this looks like a contraction function but it is a stronger condition. You will need to use the fact that \( |f(u) - f(v)| = |f(u) - z + z - f(v)| = |f(u) - z| + |z - f(v)| \) and implicitly apply an inductive argument (you don’t even have to bother being explicit) to get an expression for \( |f(u) - f(v)| \) based on refining the range into \( n \) small segments.

**Solution**

(i) \( x_2(s) = \frac{1}{2} \int_0^s \frac{15r}{12} dt + \frac{5s}{6} = \frac{13s}{24} + \frac{5s}{6} = \frac{43s}{432} = \frac{73s}{72} \)

(ii) \( x_3(s) = \frac{1}{2} \int_0^s \frac{73s}{72} dt + \frac{5s}{6} = \frac{73s}{144} + \frac{5s}{6} = \frac{259s}{432} = \frac{5s}{432} \)

To get the general formula for \( x_n(s) \) note that \( 12 = 2.6 \) and \( 13 = 2.6 + 1 \) while \( 72 = 2.36 = 2.6^2 \) and \( 73 = 2.6^2 + 1 \). Thus the guess is \( x_n(s) = \frac{2 \cdot 6^n + 1}{2 \cdot 6^n} \) for \( n = 1 \) and \( s \in [0, 1] \). The formula is true for \( n = 1 \) and we assume that it is true for all \( n \).

Now \( x_{n+1}(s) = \frac{1}{2} \int_0^s t \left( \frac{2 \cdot 6^n + 1}{2 \cdot 6^n} \right) t \ dt + \frac{5s}{6} \) (using the definition and the inductive hypothesis)

\[
\begin{align*}
&= \left( \frac{2 \cdot 6^n + 1}{2 \cdot 6^n} \right) \frac{5s}{6} \\
&= \frac{2 \cdot 6^n + 1}{2 \cdot 6^n} \frac{5s}{6} \\
&= \frac{2 \cdot 6^{n+1} + 5 \cdot 6^n}{2 \cdot 6^n} \\
&= \frac{2 \cdot 6^{n+1} + 1}{2 \cdot 6^n} \\
&= \frac{2 \cdot 6^n + 1}{2 \cdot 6^n}
\end{align*}
\]

This establishes that the formula is true for \( n + 1 \) and hence true for all \( n \) by induction.

Since we now know that \( x_n(s) = \frac{2 \cdot 6^n + 1}{2 \cdot 6^n} \) it is clear that \( \lim_{n \to \infty} x_n(s) = \lim_{n \to \infty} \left( 1 + \frac{1}{2 \cdot 6^n} \right) s = s \)

The diagram below demonstrates the convergence to \( x(s) = s \).

(iii) Suppose \( v > u \) and define \( x_0 = u \) and \( x_i = u + i \epsilon \) where \( \epsilon = \frac{v - u}{n} \) where \( n \) is a positive integer and \( i = 1, 2, ..., n \). The critical first step is to note that:
So the formula is true for \( n+1 \) and hence true for all \( n \geq 1 \) by induction.


### Solution

The formula is true for \( n = 1 \) since

\[
\begin{align*}
\frac{a^2 - b^2}{a-b} &= \frac{(a + b)(a - b)}{a-b} = a + b.
\end{align*}
\]

Assume the formula is true for any positive integer \( n \). Then:

\[
\begin{align*}
\frac{a^n + b^n}{a-b} &= \frac{a^{n+1} - b^{n+1}}{a-b} - \frac{a^{n-1} - b^{n-1}}{a-b} \\
&= a^{n+1} - b^{n+1} - \frac{a^n + b^n}{a-b} \\
&= a + b - \frac{a^n + b^n}{a-b} \\
&= \frac{a^{n+1} - b^{n+1}}{a-b}.
\end{align*}
\]

So the formula is true for \( n+1 \) and hence true for all \( n \geq 1 \) by induction.

\[
\begin{align*}
u_{n+1} &= \frac{a^{n+1} - b^{n+1}}{a-b} \\
&= \frac{a^{n+1}}{a^n} \left( \frac{1 - \left( \frac{b}{a} \right)^{n+1}}{1 - \left( \frac{b}{a} \right)} \right) \\
&= a \left( \frac{1 - \left( \frac{b}{a} \right)^{n+1}}{1 - \left( \frac{b}{a} \right)} \right)
\end{align*}
\]

Therefore \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} a \left( \frac{1 - \left( \frac{b}{a} \right)^{n+1}}{1 - \left( \frac{b}{a} \right)} \right) \)
\[ a \lim_{n \to \infty} \left( \frac{1 - \left(\frac{3}{2}\right)^{n+1}}{1 - \left(\frac{3}{2}\right)^{n}} \right) \]

\[ = a \text{ since } \frac{\beta}{a} < 1 \text{ and so } \left( \frac{1 - \left(\frac{3}{2}\right)^{n+1}}{1 - \left(\frac{3}{2}\right)^{n}} \right) \to 1 \text{ as } n \to \infty \]

For the last bit we assume that \( a = \beta = 0 \). Then \( u_1 = 2a \).

\[ u_2 = 2a - \frac{\alpha^2}{2a} = \frac{3a}{2} \] (using the definition of \( u_n \))

\[ u_3 = 2a - \frac{\alpha^2}{3a} = \frac{4a}{3} \]

....

I claim that \( u_n = \left( \frac{n + 1}{n} \right) \alpha \) for \( n \geq 1 \). The claim is true for \( n = 1 \) since the formula gives \( 2\alpha \). Assume the formula is true for any positive integer \( n \).

Then \( u_{n+1} = 2a - \frac{\alpha^2}{\left( \frac{n+1}{n} \right) \alpha} \) (using the definition \( u_n = 2\alpha - \frac{\alpha^2}{u_{n-1}} \))

\[ = \frac{2\alpha(n+1) - n\alpha^2}{(n+1)\alpha} \]

\[ = \left( \frac{n+2}{n+1} \right) \alpha \]

Hence the claim is true by induction for every \( n \geq 1 \).

Therefore \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} \left( \frac{n + 1}{n} \right) \alpha = \alpha \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = \alpha \)

#### 42. Working out the area of a triangle using lattice points

An \( n \times n \) integer lattice has a point at each integer valued ordered pair \((i,j)\) where \( i,j \) are integers \( \geq 0 \). Adjacent points are one unit apart.

A \( 4 \times 4 \) lattice looks like this:

```
* * * *
* * * *
* * * *
* * * *
```

If we form a right angled triangle with the diagonal forming the hypotenuse, what is the area of the triangle as a function of the number of boundary and interior points of the triangle so formed? The following diagram illuminates the difference between boundary and interior points:

```
B + + *
B B + *
B I B *
B B B B
```

B is a boundary point and I is an interior point.

For the triangle formed within an \( n \times n \) lattice it is claimed that the area is: \( A_n = I_n + \frac{B_n}{2} - 1 \) where :

\( A_n = \) area of the triangle formed within the \( n \times n \) lattice

\( B_n = \) the number of boundary points in the triangle

\( I_n = \) the number of interior points in the triangle.

For instance for the \( 4 \times 4 \) lattice the area of the triangle is \( \frac{1}{2} \times 3 \times 3 = \frac{9}{2} \). Alternatively the area is half that of the square i.e \( \frac{(4-0)^2}{2} \). The formula gives \( 1 + \frac{9}{2} - 1 = \frac{9}{2} \).

The aim is to prove the area formula by induction.

Find general expressions (ie based in the triangle formed in an \( n \times n \) lattice) for:

(1) The number of boundary points in the triangle.
(2) The number of interior points in the triangle

Use the formulas from (1) and (2) to prove the formula for $A_n$ by induction.

**Solution**

(1) There are $n$ points on the diagonal, $n - 1$ points on one base and $n - 2$ points on the other base. This totals $3n - 3$, so $B_n = 3(n - 1)$

(2) To work out the number of interior points I offer two approaches, one purely algebraic and the other more visual. The $i^{th}$ column in the triangle (see diagram below) gives rise to $n - i - 1$ interior points for $2 \leq i \leq n - 1$. The total number of interior points is thus:

$I_n = \sum_{i=2}^{n-1} (n - i - 1) = (n - 1) \sum_{i=2}^{n-1} 1 - \sum_{i=2}^{n-1} i = (n - 1) - 2(\frac{n(n-1)}{2} - 1)

= \frac{(n-1)(2n-4-2n+2)}{2}

= \frac{(n-1)(n-4)}{2}

= \frac{n^2 - 5n + 6}{2}

= \frac{(n-2)(n-3)}{2}$

A more visual approach is as follows:
The formula for the area is a linear function of boundary points and interior points. The area of the triangle in the general case is \( \frac{(n-1)^2}{2} \) so the formula must ultimately involve quadratic terms in \( n \) (and we'll come to that later), but since the area comprises little squares of area \( 1 \) and little triangles of area \( \frac{1}{2} \) it does make sense to assert that \( A_n = f(B_n, I_n) = u B_n + v I_n + w \) for some constants \( u, v, w \). Using our formulas for \( I_n \) and \( B_n \), \( A_n = u \frac{3(n-1)}{2} + v \frac{(n-2)(n-3)}{2} + w \)

The smallest value for \( n \) is 3 and the area of the triangle formed in that lattice is \( \frac{1}{2} \times 2 \times 2 = 2 \). Similarly for \( n = 4 \) and 5 the areas are respectively \( \frac{1}{2} \times 3 \times 3 = 9 \) and \( \frac{1}{2} \times 4 \times 4 = 8 \). The formulas for \( A_3 \) etc are as follows:

\[
A_3 = 6u + 0v + w = 2 \\
A_4 = 9u + v + w = \frac{9}{2} \\
A_5 = 12u + 3v + w = 8
\]

In matrix form this is

\[
\begin{pmatrix}
6 & 0 & 1 \\
9 & 1 & 1 \\
12 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
= 
\begin{pmatrix}2 \\
\frac{9}{2} \\
8
\end{pmatrix}
\]

The inverse of the matrix is

\[
\begin{pmatrix}
-2 & 1 & -1 \\
1 & -2 & 1 \\
5 & -6 & 2
\end{pmatrix}
\begin{pmatrix}u \\
v \\
w
\end{pmatrix}
= 
\begin{pmatrix}\frac{1}{2} \\
\frac{1}{2} \\
-1
\end{pmatrix}
\]

If you have shamefully forgotten how to invert a 3 x 3 matrix, here is the process. First, to work out the determinant of the matrix we can expand by the minors along the first row because one of the row elements is zero. This saves work. Thus \( \det
\begin{pmatrix}
6 & 0 & 1 \\
9 & 1 & 1 \\
12 & 3 & 1
\end{pmatrix}
= 6 \cdot \det
\begin{pmatrix}
9 & 1 \\
12 & 3
\end{pmatrix}
- 0 \cdot \det
\begin{pmatrix}
9 & 1 & 1 \\
12 & 3 & 1
\end{pmatrix}
+ 1 \cdot \det
\begin{pmatrix}
9 & 1 \\
12 & 3
\end{pmatrix}
= 6 \cdot -2 + 1 \cdot 15 = 3
\]

Next we need to find the adjoint of the matrix. The adjoint of an \( n \times n \) matrix \( A \) whose \( (i,j) \) entry \( \text{adj}(A)_{ij} \) is \( (-1)^{i+j} \det A_{ij} = \alpha_{ij} \), where \( A_{ij} \) is the matrix formed by crossing out the \( i \)th row and the \( j \)th column. Thus \( \text{adj}(A) = (\alpha_{ij})^t \). In the 2 x 2 case \( \text{adj}(\begin{pmatrix}a & b \\
c & d\end{pmatrix}) = \begin{pmatrix}d & -b \\
-c & a\end{pmatrix} \).

In our case \( \text{adj}
\begin{pmatrix}
6 & 0 & 1 \\
9 & 1 & 1 \\
12 & 3 & 1
\end{pmatrix}
= 
\begin{pmatrix}
-2 & 3 & 15 \\
3 & -6 & -18 \\
-1 & 3 & 6
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
= 
\begin{pmatrix}
-2 & 3 & -1 \\
3 & -6 & 3 \\
15 & -18 & 6
\end{pmatrix}
\]

Hence the inverse we want is\( \frac{1}{2}
\begin{pmatrix}
-2 & 3 & -1 \\
3 & -6 & 3 \\
15 & -18 & 6
\end{pmatrix}
\begin{pmatrix}u \\
v \\
w
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{2} \\
\frac{1}{2} \\
-1
\end{pmatrix}
\]

Hence we get \( A_n = I_n + \frac{1}{2} B_n - 1 \) as asserted. But is it correct? Our base case is \( n = 3 \) and the formula is true for this value since \( A_3 = 0 + \frac{3 \times 2}{2} - 1 = 2 \) and the area of the 3-lattice triangle is \( \frac{(3-1)^2}{2} = 2 \). Let's assume \( A_n \) is true for any \( n \geq 3 \). Now consider an \( (n+1) \) lattice and the triangle generated by it. The total area of the new triangle is \( A_n + \text{the extra trapezium} \) (see diagram).
\[ A_{n+1} = A_n + \frac{2n-1}{2} \]

\[ = I_n + \frac{B_{n-1}}{2} + \frac{2n-1}{2} \] using the induction hypothesis

\[ = \frac{n(n-2)(n-3)}{2} + \frac{3(n(n-1)}{2} + \frac{2n-3}{2} \]

\[ = \frac{n^2 - 5n + 6 + 3n - 3 + 2n - 3}{2} \]

\[ = \frac{n^2}{2} \] which is the area of the required triangle.

Note that \( A_{n+1} = I_{n+1} + \frac{1}{2} B_{n+1} - 1 = \frac{(n-1)(n-2)}{2} + \frac{3n}{2} - 1 \)

\[ = \frac{n^2 - 3n + 2 + 3n - 2}{2} \]

\[ = \frac{n^2}{2} \]

Thus the formula is true for \( n + 1 \). Remember \( A_n = \frac{(n - 1)^2}{2} \)

This approach can be generalised as is done by Richard E Crandall, "Projects in Scientific Computation", Springer-Telos, 1996, page 55. The problem he poses is to develop software to find polygonal areas using the lattice area formula where all of the polygon's vertices lie on integer coordinates. The formula is \( A = I + \frac{B}{2} - 1 \) where \( I \) is the number of strictly interior points, \( B \) is the number of boundary points (vertices and any other lattice points precisely skewered by the sides of the polygon). Prove the formula by induction (it helps to consider simple general assemblies of squares or triangles to get the drift of the theory\(^b\)). The following diagram gives the gist of what Crandall was getting at:

Given the requirement that every vertex lies on a lattice point it is possible to represent the total area as the sum of squares and right
43. Volume of a unit ball in n dimensions

In Problem 32 we encountered the Gamma function. In this problem we will see how it features in the generalisation of volumes in n dimensions and how induction is useful in proving the general formula. We will work with unit balls in \( \mathbb{R}^n \) since it is a basic, indeed crucial, concept of measure theory (and hence integration theory) that the measure or volume of a set is invariant under translations. Translation invariance means that if we have a cube of side length \( r \) centred at \((0,0,0)\) and we centre it at \((x,y,z)\), the volume does not change. You would have implicitly assumed this in any event. There is a form of relative dilation invariance as follows. If we take the volume of a unit cube (which can be centred anywhere since volume is translation invariant) and scale up the sides by \( r \), the volume scales by \( r^n \) (we would end up with a cube of volume \( r^n \)). Note that if you have a square, say, and you double each side the area increases by a factor of 4. Volume being the area times a length would then be \( 8 \) times the original length cubed. Thus when we go to n dimensional space the concept is that the volume of an n dimensional “ball” would be \( v_n r^n \) where \( r \) is the "radius" of the ball and \( v_n \) is the volume of the unit ball in \( \mathbb{R}^n \). In effect the volume of the unit ball is scaled by \( r^n \). This concept is important in what follows. Note that when we talk about volumes generally lengths and areas count as (degenerate) "volumes" for the sake of consistency.

There is a relationship between volume of an n-dimensional unit ball and an n-dimensional unit sphere (note that the unit ball is the set of all points enclosed by the unit sphere which in turn is the set of points at distance 1 from a fixed point). Start with the unit ball (of volume \( v_n \)) and then enlarge it a bit by \( \epsilon > 0 \). The enlarged ball has volume \( v_{n+\epsilon} = (1 + \epsilon)^n v_n \) because the volume scales by \( (1 + \epsilon)^n \). The volume between the large ball and the small one is \( v_{1+\epsilon} - v_{n+\epsilon} = \{(1 + \epsilon)^n - 1\} v_n \). If \( s_n \) is the surface area of the unit ball then this volume is approximately that are times the thickness \( \epsilon \) ie \( \epsilon s_n \, \{(1 + \epsilon)^n - 1\} v_n \).

This suggests that in the limit \( s_n = \lim_{\epsilon \to 0} \frac{(1 + \epsilon)^n - 1}{\epsilon} \) \( v_n = \lim_{\epsilon \to 0} \frac{1 + \epsilon + \frac{(n(1 + \epsilon))}{2} + \frac{(n(1 + \epsilon))^2}{3} + \ldots + 1}{\epsilon} v_n \). If \( s_n \) is the surface area of the unit ball then this volume is approximately that are times the thickness \( \epsilon \) ie \( \epsilon s_n \, \{(1 + \epsilon)^n - 1\} v_n \).

To work out the volume of an n dimensional unit ball we can't actually visualise (at least I can't!) the physical construction. What one has to do is generalise the concepts of lower order dimensions which we can visualise and then rely upon the dilation principle discussed above to extend the process. There are various ways of arriving at the required volume and they are usually taught in multivariate calculus courses. One common approach is to concoct an n dimensional integral which involves \( e^{-r^2} \) and it looks like this:

\[ I = \int_{\mathbb{R}^n} e^{-r^2} \, dx_1 dx_2 \ldots dx_n \]  

The integral has the form \( \pi^{n/2} \). One then does an integral by breaking \( \mathbb{R}^n \) into “spherical” shells and when this is done the gamma function emerges, essentially because a change of variable is used to get rid of the \( -r^2 \) in the exponential in favour of \( -t \). You end up with an expression that looks like this:

\[ I = \frac{1}{2} \text{Volume } (n - 1 \text{ dimensional ball}) \times \int_0^{\infty} e^{-t} t^{n/2 - 1} \, dt \]

Thus the gamma function emerges. This approach can be found in Barton Zwiebach, "A First Course in String Theory", Cambridge University Press, 2004, pages 46-49 and many other places. A short proof which follows a similar type of approach can be found here: [http://www.math.uu.se/~svante/papers/sjk2.ps](http://www.math.uu.se/~svante/papers/sjk2.ps)

The problem with the above type of proof is the introduction of the exponential factor which has no analog in the lower dimensional cases. Thus when you work out the volume of a sphere using basic calculus techniques you don’t get an \( e^{-r^2} \) factor. You simply work out the volume of an elementary disk and then sum appropriately - no mention of \( e^{-r^2} \). This is the approach adopted in Harley Flanders, "Differential Forms with Applications to the Physical Sciences", Dover Publications, 1989, pages 74-75. Harley Flanders is an American mathematician who has written several influential textbooks. If you adopt that approach where does the gamma function come from? That is what this problem is about.

What Flanders does is this. He asserts that the volume of an n-dimensional unit ball can be found by integrating over slabs via this recurrence relation:

\[ V_n = \int_{-1}^1 (1 - x^2)^{n/2 - 1} V_{n-1} \, dx \]
\[ V_{n-1} \text{ where } J_n = \int_1^1 (1 - x^2)^{\frac{n-1}{2}} \, dx \]

Now to see why the formula for \( V_n \) makes sense consider the case when \( n = 2 \). The 2 dimensional "volume" is an area. You would then have \( V_2 = \int_1^1 \sqrt{1 - x^2} \, dV_1 \). What is the "volume" of a 1 dimensional unit ball ie \( V_1 \)? It must be 2 ie the length from -1 to 1. Thus \( V_2 = 2 \int_1^1 \sqrt{1 - x^2} \, dx \). Does this make sense? It does, for consider the diagram below:

The volume (ie area) \( V_2 \) is indeed \( 2 \int_1^1 \sqrt{1 - x^2} \, dx = \pi \). See, no \( e^{-x^2} \) factor up my sleeve!. Now to generalise this process we take an (n-1) dimensional slab which has volume \( V_{n-1} \) multiply by the slab width \( dx \) and finally multiply by the "height" \( \left( \sqrt{1 - x^2} \right)^{n-1} \). The height is the usual radial height of \( \sqrt{1 - x^2} \) raised to the dimension of the space in which the sphere is "immersed" lowered by 1. In the simple example above the circle is "immersed" in \( \mathbb{R}^2 \) so the height is \( \left( \sqrt{1 - x^2} \right)^{2-1} = \sqrt{1 - x^2} \). Thus when we do \( V_3 \) we get:

\[ V_3 = \int_1^1 (1 - x^2)^{\frac{3}{2}} V_2 \, dx = \pi \int_1^1 (1 - x^2) \, dx = \pi \left[ x - \frac{x^3}{3} \right]_1^1 = \frac{4\pi}{3} \]

Note that this is equivalent in traditional calculus methods to getting the volume of an elementary disk being \( \pi y^2 \, dx \) where \( 1 = x^2 + y^2 \) and then integrating from \( x = -1 \) to \( x = 1 \).

So the formula \( V_n = \int_1^1 (1 - x^2)^{\frac{n-1}{2}} \, V_{n-1} \, dx \) does make sense.

Flanders then derives the following recurrence relation by doing integration by parts once:

\[ J_n = \frac{n-1}{n} J_{n-2} \]

He then says "these recurrence formulae lead to the standard result : \( V_n = \frac{\pi^\frac{n}{2}}{\Gamma\left(\frac{n+1}{2}\right)} \)." So the gamma function finally raises its ugly head! He leaves out the details and this problem is about filling them in.

**What you have to establish is that** \( J_n = \frac{n-1}{n} J_{n-2} \) and then that \( V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \).

It helps to establish the following propositions as building blocks:

(i) \( J_{2k} = \frac{(2k-1)(2k-3)\ldots \cdot 3 \cdot 1}{2^k k!} \pi = \frac{(2k-1)(2k-3)\ldots \cdot 3 \cdot 1}{2^k k!} \pi \) for positive integers \( k \) 1.

(ii) \( J_{2k-1} = \frac{(2k-2)(2k-4)\ldots \cdot 2 \cdot 0}{(2k-1)(2k-3)\ldots \cdot 3} J_1 = \frac{2(2k-1)!}{(2k-1)(2k-3)\ldots \cdot 3} \) for positive integers \( k \) 2.

**Solution**

To establish \( J_n = \frac{n-1}{n} J_{n-2} \) where \( J_n = \int_1^1 (1 - x^2)^{\frac{n-1}{2}} \, dx \) just do one integration by parts as follows:

Let \( u = (1 - x^2)^{\frac{n-1}{2}} \) and \( dv = dx \) so \( v = x \)

\[ J_n = \int_1^1 u \, dv = uv\big|_1^1 - \int_1^1 v \, du = -\int_1^1 x \cdot \frac{(n-1)}{2} (1 - x^2)^{\frac{n-1}{2}} (-2x) \, dx \]

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\[= \int_1^0 x^2 (n - 1) (1 - x^{2^{n-1}}) \, dx \]

\[= \int_1^0 (n - 1)(1 - x^2) \, dx + \int_1^0 (n - 1)(1 - x^{2^{n-1}}) \, dx \]

\[= -(n - 1) \int_1^0 (1 - x^2) \, dx + (n - 1) J_{n-2} \]

\[-(n - 1) J_n + (n - 1) J_{n-2} \]

Therefore \( J_n = (n - 1) J_{n-2} \) and so \( J_n = \frac{n-1}{n} J_{n-2} \)

We know that \( V_n = V_{n-1} J_n \) so we proceed iteratively as follows:

\( V_{n-1} = V_{n-2} J_{n-1} \) so \( V_n = V_{n-2} J_{n-1} \cdot J_n = J_{n-1} J_n \cdot J_{n-2} \cdot J_1 \). This can be established inductively. First, the base case of \( n = 2 \) is true since \( V_2 = V_1 J_2 \) and \( V_1 = 2 \) (as we established earlier). As usual we assume the formula is true for any \( n \). Now \( J_2 = \int_1^0 \sqrt{1 - x^2} \, dx = \frac{\pi}{2} \) . If this integral doesn't trip easily from the tongue simply let \( x = \cos u \) so \( dx = -\sin u \, du \). The integral becomes:

\[-\frac{1}{2} \int_0^1 \sin^2 u \, du = \int_0^1 \sin^2 u \, du = \frac{1}{2} \int_0^1 (1 - \cos 2u) \, du \]

\[= \frac{1}{2} u - \frac{1}{2} \sin 2u \bigg|_0^1 \]

\[= \frac{\pi}{2} \]

Thus \( V_2 = 2 \times \frac{\pi}{2} = \pi \) which we also established above by traditional methods. Now \( V_{n+1} = V_n J_{n+1} = J_{n+1} J_n J_{n-1} \cdot J_1 \) using the induction hypothesis. So the formula is true for \( n+1 \) and hence true for all \( n \).

(i) Prove that \( J_{2k} = -\frac{(2k-1)(2k-3)\ldots 3}{(2k)(2k-2)\ldots 4} \frac{\pi}{2^{2k}} \) for positive integers \( k \geq 1 \). From the recurrence relation \( J_{2k} = \frac{2k-1}{2k} J_{2k-2} = \frac{2k-1}{2k} \frac{2k-3}{2k-2} J_{2k-4} = \ldots = \frac{(2k-1)(2k-3)\ldots 3}{(2k)(2k-2)\ldots 4} J_4 = \frac{(2k-1)(2k-3)\ldots 3}{(2k)(2k-2)\ldots 4} \frac{\pi}{2^{2k}} \)

Now note that in the numerator and denominator there are \( k \) terms eg \( J_6 = \frac{5}{6} J_4 = \frac{5}{6} \frac{5}{6} J_2 = \frac{5}{6} \frac{5}{6} \frac{1}{2} \pi = \frac{5}{6} \frac{5}{6} \frac{1}{2} \pi \)

(ii) Prove that \( J_{2k-1} = \frac{(2k-2)(2k-4)\ldots 2}{(2k)(2k-2)\ldots 4} J_{2k-3} = \frac{(2k-2)(2k-4)\ldots 2}{(2k)(2k-2)\ldots 4} \frac{\pi}{2^{2k-1}} \) for positive integers \( k \geq 2 \).

The proof of this follows the same approach as in (i) above.

\( J_{2k-1} = \frac{2k-2}{2k-1} J_{2k-3} = \frac{(2k-2)(2k-4)\ldots 2}{(2k-1)(2k-3)\ldots 3} \frac{\pi}{2^{2k-1}} \) Here \( k \geq 4 \) and there are \( 4-1=3 \) factors.

Hence \( J_{2k-1} = \frac{(2k-2)(2k-4)\ldots 2}{(2k-1)(2k-3)\ldots 3} J_1 = \frac{2^k(k-1)!}{(2k)(2k-2)\ldots 4} \) since \( J_1 = 2 \). Note that \( J_n = \int_1^0 (1 - x^2) \frac{x^{2n-1}}{2^{2n-1}} \, dx \) therefore \( J_1 = \int_0^1 dx = 2 \)

Again, we can do a quick inductive proof to convince ourselves that this is correct. The formula is true for the base case \( k = 2 \) since \( J_3 = \frac{2^3}{3} \pi = \frac{4}{3} \pi \) and we know that \( J_2 = \frac{2^2}{2} \pi = \pi \)

Consider \( J_{2k+1} = \frac{2k+1}{2k+2} J_{2k+1} = \frac{(2k+1)(2k-1)!}{(2k+1)(2k-1)!} \frac{\pi}{2^{2k+1}} \) which is the right structure. Hence the formula is established as correct for all \( k \) by induction.

**Getting the expression for the volume...**

Let's start with even dimensions.

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Now if you play around with the product \( V_n = J_n J_{n-1} \ldots J_2 J_1 \) you will notice that when \( n = 2k \) the product is \( \frac{\pi^2}{4k} \). Again, this involves an inductive style of argument. Let’s do a few cases to see the pattern. Note that \( J_1 = 2, J_2 = \frac{\pi}{2}, J_3 = \frac{2}{3} J_1 = \frac{4}{3} \) and \( J_4 = \frac{3}{4} J_2 = \frac{3\pi}{8} \) etc.

\[
J_1 J_2 = 2 \times \frac{\pi}{2} = \frac{\pi}{1} 
\]

\[
J_1 J_2 J_3 J_4 = \pi \times \frac{2}{3} J_1 \times \frac{3}{4} J_2 \text{ using the recurrence formula for } J_n 
\]

\[
= \pi \times \frac{2}{3} \times 2 \times \frac{3}{4} \times \frac{\pi}{2} 
= \frac{\pi^2}{2!} 
\]

\[
J_1 J_2 J_3 J_4 J_5 J_6 = \frac{\pi^2}{2!} \times \frac{4}{5} J_3 \times \frac{5}{6} J_4 
\]

\[
= \frac{\pi^2}{2!} \times \frac{4}{5} \times \frac{4}{3} \times \frac{5}{6} \times \frac{3\pi}{8} 
= \frac{\pi^3}{6} 
= \frac{\pi^3}{3!} 
\]

Our assertion is that \( V_{2k} = \frac{\pi^k}{k!} = \frac{\pi^k}{\Gamma(k+1)} \). Now this is true for \( k = 1 \) since \( V_2 = \pi \) (already established above) and \( \text{RHS} = \frac{\pi}{\Gamma(1)} = \frac{\pi}{1} = \pi \).

See Problem 32 for the properties of the gamma function.

So now consider \( V_{2k+2} = J_{2k+2} J_{2k+1} J_{2k} \ldots J_2 J_1 \)

\[
= \frac{2^{k+1}}{2k+2} J_{2k+1} J_{2k-1} \frac{\pi^k}{k!} \text{ using the induction hypothesis} 
\]

\[
= \frac{2k+1}{2k+2} \frac{2k}{2k+1} \frac{2^{k-1}}{(2k-1)(2k-3) \ldots 3} \pi \frac{2k}{2k+1} \frac{2^{k-1}}{(2k-1)(2k-3) \ldots 3} \frac{\pi^k}{k!} \text{ using (i) and (ii) above.} 
\]

\[
= \frac{\pi^{k+1}}{(k+1)!} \text{ since } \Gamma \left( \frac{2k+2}{2} \right) = (k+1)! \text{ (already proved by induction in Problem 32)} 
\]

So the formula is true for \( k+1 \) and hence true for all \( k \)

Now consider odd cases of \( n = 2k+1 \). So \( V_{2k+1} = J_1 \ldots J_k J_{2k+1} = \frac{\pi^k}{k!} \frac{2^{k+1}}{(2k+1)(2k-1)(2k-3) \ldots 3} \) using (ii) with \( 2k+1 \) instead of \( 2k-1 \)

\[
= \frac{2^{k+1} \pi^k}{(2k+1)(2k-1)(2k-3) \ldots 3} 
\]

We want to show that \( \frac{2^{k+1} \pi^k}{(2k+1)(2k-1)(2k-3) \ldots 3} = \frac{\pi^k}{(k+1)!} \). Expanding \( \Gamma \left( \frac{2k+1}{2} + 1 \right) = \frac{2k+1}{2} \Gamma \left( \frac{2k+1}{2} \right) = \frac{2k+1}{2} \Gamma \left( \frac{2k-1}{2} + 1 \right) = \frac{2k-1}{2} \Gamma \left( \frac{2k-1}{2} \right) \)

\[
\Gamma \left( \frac{2k-1}{2} \right) = \ldots = \frac{(2k+1)(2k-1)(2k-3) \ldots 3}{2^{k+1} \pi^k} 
\]

Thus the assertion is that \( \Gamma \left( \frac{2k+1}{2} + 1 \right) = \frac{(2k+1)(2k-1)(2k-3) \ldots 3}{2^{k+1} \pi^k} \sqrt{\pi} \) and this is amenable to an inductive style of proof. The assertion is true for the base case of \( k = 1 \) since \( \Gamma \left( \frac{3}{2} + 1 \right) = \frac{3}{2} \Gamma \left( \frac{1}{2} \right) = \frac{3}{2} \Gamma \left( \frac{1}{2} + 1 \right) = \frac{3}{2} \Gamma \left( \frac{1}{2} \right) \)

\[
= \frac{3 \sqrt{\pi}}{2^2} . \text{ The RHS of the assertion gives } \frac{3}{2} \sqrt{\pi} \text{ and so the base case is true. Now consider the situation for } k+1: 
\]

\[
\Gamma \left( \frac{2k+3}{2} + 1 \right) = \frac{2k+3}{2} \Gamma \left( \frac{2k+3}{2} \right) = \frac{2k+3}{2} \Gamma \left( \frac{2k+1}{2} + 1 \right) 
\]

\[
= \frac{(2k+1)(2k+3)(2k-1)(2k-3) \ldots 3}{2^{k+1} \pi^k} \sqrt{\pi} \text{ using the induction hypothesis.} 
\]
successive differences as follows:

\[ (AB)f = A(Bf) \]

4. \( (A + B)f = Af + Bf \) and \( (A - B)f = Af - Bf \)

There is no confusion.

Let's take a classic operator - the difference operator \( \Delta \) which is defined as acting on functions \( f(x) \) as follows:

\[ \Delta f(x) = f(x+h) - f(x) \]

where \( h \) is usually a positive number which is the "difference interval". It is a linear operator. You can take successive differences as follows:

\[ \Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[ f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x) \]

\[ \Delta^3 f(x) = \Delta[\Delta^2 f(x)] = \Delta[ f(x+2h) - f(x+h) - (f(x+h) - f(x))] = f(x+2h) - 2f(x+h) + f(x) \]

The \( n \)-th order difference operator is defined inductively as \( \Delta^n f(x) = \Delta[\Delta^{n-1} f(x)] \)

The usual algebraic properties are also relevant to operators, namely:

1. \( A = B \) iff \( Af = Bf \)
2. There is an identity operator \( I \) such that \( If = f \). In what follows \( I \) will use "0" to represent this operator and, again, the context will ensure there is no confusion.
3. There is a null operator \( 0 \) such that \( 0f = 0 \) for every \( f \). I will use "0" to represent this operator and, again, the context will ensure there is no confusion.
4. \( (A + B)f = Af + Bf \) and \( (A - B)f = Af - Bf \)
5. \( (AB)f = A(Bf) \)
6. \( A(1+q) = Af + Ag \) and \( A(\beta f) = \beta Af \) where \( \beta \) is a constant. Thus \( A \) is a linear operator if this condition is satisfied.
7. If \( A(0) = 0 \) for any \( f \) then we have that \( AB = 0 \) and B is called the inverse of \( A \). Thus \( B = A^{-1} = \frac{1}{A} \)

The usual algebraic properties are also relevant to operators, namely:

8. Commutativity for sums: \( A + B = B + A \)
9. Associativity for sums: \( A + (B + C) = (A + B) + C \)
10. Commutativity for products: \( AB = BA \)
11. Associativity for products: \( A(BC) = (AB)C \)
12. Distributive law: \( A(B+C) = AB + AC \)

You have to check that your operator obeys these rules before you perform various manipulations. If commutativity does not hold this statement will be false \( AB \neq BA = 0 \).

Let's take a classic operator - the difference operator \( \Delta \) which is defined as acting on functions \( f(x) \) as follows:

\[ \Delta f(x) = f(x+h) - f(x) \]

where \( h \) is usually a positive number which is the "difference interval". It is a linear operator. You can take successive differences as follows:

\[ \Delta^2 f(x) = \Delta[\Delta f(x)] = \Delta[ f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x) = f(x+2h) - f(x+h) - (f(x+h) - f(x)) = f(x+2h) - 2f(x+h) + f(x) \]

The usual algebraic properties are also relevant to operators, namely:

1. \( A + B = B + A \)
2. \( (A + B)f = Af + Bf \)
3. \( (A - B)f = Af - Bf \)
4. \( (AB)f = A(Bf) \)
5. \( (AB + AC)f = A(Bf + Cf) \)
6. \( (AB + AC)f = A(Bf) + A(Cf) \)
7. \( (AB + AC)f = (A + C)(Bf) \)

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44. Difference and translation operators

In actuarial theory, students have traditionally studied difference and translation operators in the context of the theory of finite differences and difference equations. This problem deals with some basic theory of operators and how you can use induction to prove some properties. We can think of squaring, taking the derivative or performing an integral as operators to have all the "nice" properties such as follows (where \( A \) and \( B \) are two operators and \( f \) is some function):

1. \( A = B \) iff \( Af = Bf \)
2. There is an identity operator \( I \) such that \( If = f \). In what follows \( I \) will use "0" to denote the identity operator and the context will ensure that there is no confusion.
3. There is a null operator \( 0 \) such that \( 0f = 0 \). I will use "0" to denote this operator and, again, the context will ensure there is no confusion.
4. \( (A + B)f = Af + Bf \)
5. \( (AB)f = A(Bf) \)
6. \( A(1+q) = Af + Ag \) and \( A(\beta f) = \beta Af \) where \( \beta \) is a constant. Thus \( A \) is a linear operator if this condition is satisfied.
7. If \( A(0) = 0 \) for any \( f \) then we have that \( AB = 0 \) and \( B \) is called the inverse of \( A \). Thus \( B = A^{-1} = \frac{1}{A} \)

The usual algebraic properties are also relevant to operators, namely:

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In a similar fashion one can define a linear translation operator as follows: \( Ef(x) = f(x + h) \). Applying this operator twice you get: \( E^2 f(x) = E[f(x + h)] = f(x + 2h) \). The \( n \)-th order version \( E^n \) is defined inductively as \( E^n f(x) = f(x + nh) \).

### Problems

Show that:

(i) \( \Delta = E - 1 \)

(ii) \( E = 1 + \Delta \)

(iii) \( \Delta^2 = (E - 1)^2 = E^2 - 2E + 1 \); and more generally

(iv) \( \Delta^n = (E - 1)^n = E^n - \binom{n}{0} E^{n-1} + \binom{n}{2} E^{n-2} + \ldots + (-1)^n \) where \( \binom{n}{r} = \frac{n!}{(n-r)!r!} \)

(v) Show that

(vi) Deriving Newton’s difference formula: If \( r > 1 \), define \( (1 + \Delta)^r u_n \) inductively as \( (1 + \Delta)^r u_n \) for all \( n \).

(i) Show that \( (1 + \Delta)^r u_n = \binom{r}{1} \Delta u_n + \binom{r}{2} \Delta^2 u_n + \ldots + \binom{r}{r} \Delta^r u_n \)

(ii) Show that \( u_{n+1} = (1 + \Delta)^n u_n \)

This is Newton’s difference formula

(vii) If \( \Delta u_n = u_{n+1} - u_n \) show that the \( u_n \) for \( 1 \leq n \) are all equal if and only if \( \Delta u_1 = \Delta u_2 = \ldots = \Delta u_{n-1} = 0 \). In other words the \( u_n \) are independent of \( n \).

(viii) Let \( u_1, u_2, \ldots \) be a given sequence and suppose we construct \( v_1, v_2, \ldots \) such that \( \Delta v_n = u_n \) for all \( n \). Then if \( S_n = \sum_{r=1}^{n} u_r \) show that \( S_n = (1 + \Delta)^n - 1) v_1 \)

### Solutions

(i) \( \Delta f(x) = f(x + h) - f(x) = Ef(x) - 1f(x) = (E - 1) f(x) \)

(ii) \( Ef(x) = f(x + h) = Ef(x + h) = (E - 1) f(x) \)

(iii) \( \Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta[(f(x + h) - f(x)] = f(x + 2h) - f(x + h) - [f(x + h) - f(x)] = f(x + 2h) - 2f(x + h) + f(x) \)

\( (E^2 - 2E + 1) f(x) = E^2 f(x) - 2 f(x) + 1 f(x) \)

\( = f(x + 2h) - 2f(x + h) + f(x) \)

\( = \Delta^2 f(x) \)

(iv) From (i) and (ii) we know that the proposition is true for \( n = 1 \) and 2. As our induction hypothesis assume that \( \Delta^n = (E - 1)^n = E^n - \binom{n}{0} E^{n-1} + \binom{n}{2} E^{n-2} + \ldots + (-1)^n \) where \( \binom{n}{r} = \frac{n!}{(n-r)!r!} \)

\( \Delta^{n+1} f(x) = \Delta[ E^n f(x) - \binom{n}{0} E^{n-1} f(x) + \binom{n}{2} E^{n-2} f(x) + \ldots + (-1)^n f(x) ] \)

\( = \Delta \left[ f(x + nh) - \binom{n}{1} f(x + (n - 1) h) + \binom{n}{2} f(x + (n - 2) h) + \ldots + (-1)^n f(x) \right] \)

\( = f(x + (n+1) h) - f(x + nh) - \binom{n}{1} [ f(x + nh) - f(x + (n - 1) h) ] + \binom{n}{2} [ f(x + (n - 1) h) - f(x + (n - 2) h) ] + (-1)^n [ f(x + h) - f(x) ] \)

using the nice linear properties of \( \Delta \)

Now Pascal’s Identity can be used to good effect here since \( \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1} \)

So \( \Delta^{n+1} f(x) = f(x + (n+1) h) - \left( \binom{n}{0} + \binom{n}{1} \right) f(x + nh) + \left( \binom{n}{1} + \binom{n}{2} \right) f(x + (n-1) h) - \ldots + (-1)^{n+1} f(x) \)
\[ f(x + (n+1)h) - \left( \frac{n + 1}{2} \right) f(x + nh) + \left( \frac{n + 1}{2} \right) f(x + (n-1)h) - \ldots - (-1)^{n+1} f(x) \]

This establishes that the formula is true for \( n+1 \) and hence true for all \( n \).

(v) Show that \((1 + \Delta)^n \Delta u_k = \Delta (1 + \Delta)^n u_k \)

Perform an induction on \( n \). When \( n = 1 \) we have that \((1 + \Delta) \Delta u_k = \Delta u_k + \Delta^2 u_k \)

Also \( (1 + \Delta) u_k = (\Delta + \Delta^2) u_k = -\Delta u_k + \Delta^2 u_k \)

As the induction hypothesis suppose the formula holds for any \( n \). Then:

\[
(1 + \Delta)^{n+1} \Delta u_k = (1 + \Delta) \{(1 + \Delta)^n \Delta u_k \}
\]

\[
= (1 + \Delta) (1 + \Delta)^n u_k \text{ using the induction hypothesis}
\]

\[
= \Delta (1 + \Delta) (1 + \Delta)^n u_k \text{ using the fact that the formula is true for } n = 1 \text{ ie } (1 + \Delta) \Delta = \Delta (1 + \Delta)
\]

\[
= \Delta (1 + \Delta)^{n+1} u_k \text{ thus establishing that the formula is true for } n+1 \text{ and hence true generally by induction}
\]

(vi) (i) Show that \((1 + \Delta)^r u_n = (1 + \Delta) \{(1 + \Delta)^{r-1} u_n \}

When \( r = 2 \) we have \((1 + \Delta)^2 u_n = (1 + \Delta) \{(1 + \Delta) u_n \}

\[
= (1 + \Delta) \left( u_n + u_{n+1} - u_k \right)
\]

\[
= (1 + \Delta) u_{n+1}
\]

\[
= u_{n+1} + \Delta u_{n+1}
\]

\[
= u_{n+1} + u_{n+2} - u_n + 1
\]

\[
= u_{n+2}
\]

\[
(1 + \left( \frac{2}{1} \right) \Delta + \left( \frac{2}{2} \right) \Delta^2) u_n = u_n + 2\Delta u_n + \Delta^2 u_n
\]

\[
= u_n + 2 (u_{n+1} - u_n) + \Delta (u_{n+1} - u_n)
\]

\[
= 2 u_{n+1} - u_n + u_{n+2} - u_{n+1} = - (u_{n+1} - u_n)
\]

\[
= u_{n+2}
\]

Hence formula is true for the base case \( r = 2 \).

Assume as the induction hypothesis that it is true for any \( r \). Then:

\[
(1 + \Delta)^{r+1} u_n = (1 + \Delta) \{(1 + \Delta)^r u_n \}
\]

\[
= (1 + \Delta) \left\{ 1 + \left( \frac{r}{1} \right) \Delta + \left( \frac{r}{2} \right) \Delta^2 + \ldots + \left( \frac{r}{k} \right) \Delta^k + \ldots + \left( \frac{r}{k} \right) \Delta^{k+1} + \ldots + \left( \frac{r}{k} \right) \Delta^{k+r} \right\} u_n
\]

\[
= (1 + \left( \frac{r}{1} \right) \Delta + \left( \frac{r}{2} \right) \Delta^2 + \ldots + \left( \frac{r}{k} \right) \Delta^k + \ldots + \left( \frac{r}{k} \right) \Delta^{k+1} + \ldots + \left( \frac{r}{k} \right) \Delta^{k+r} \right\} u_n
\]

\[
= 1 + \left\{ 1 + \left( \frac{r}{1} \right) + \left( \frac{r}{2} \right) + \ldots + \left( \frac{r}{k} \right) \right\} \Delta^2 + \ldots + \left( \frac{r}{k} \right) \Delta^k + \ldots + \left( \frac{r}{k} \right) \Delta^{k+r} \right\} u_n
\]

\[
= 1 + \left\{ \left( \frac{r + 1}{1} \right) \Delta + \left( \frac{r + 1}{2} \right) \Delta^2 + \ldots + \left( \frac{r + 1}{k} \right) \Delta^k + \ldots + \left( \frac{r + 1}{k} \right) \Delta^{k+r} \right\} u_n \text{ using Pascal's Identity.}
\]

Thus the formula is true for \( r + 1 \) and is true for all \( r > 1 \) by induction.

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Thus, from the above, is independent of \( n \). In particular we can say that

\[
\Delta^2 u_1 + \ldots + \left( \frac{n}{1} \right) \Delta u_1 + u_1 = \Delta^0 u_1
\]

This is Newton's difference formula.

The starting point is this:

\[
u_{n+1} = u_n + (u_{n+1} - u_n) = u_n + \Delta u_n = (1 + \Delta) u_n
\]

We can proceed inductively from this starting point as follows:

\[
u_{n+1} = (1 + \Delta) u_n = (1 + \Delta)^2 u_{n-1} = \ldots = (1 + \Delta)^n u_1
\]

Note that the sum of the indices is always \( n+1 \) (i.e., \( n+1 = 2 + n-1 \) etc).

The formula is true for \( n = 1 \) since \( u_2 = u_1 + (u_2 - u_1) = (1 + \Delta) u_1 \)

Assume as our induction hypothesis that the formula is true for any \( n \) i.e. \( u_{n+1} = (1 + \Delta)^n u_1 \) Then:

\[
u_{n+2} = u_{n+1} + (u_{n+2} - u_{n+1}) = u_{n+1} + \Delta u_{n+1} = (1 + \Delta) u_{n+1}
\]

\[
= (1 + \Delta)(1 + \Delta)^n u_1 \quad \text{using the induction hypothesis}
\]

\[
= (1 + \Delta)^{n+1} u_1
\]

This establishes that the formula is true for \( n+1 \) and hence true for all \( n \) by induction.

The binomial expansion of \( (1 + \Delta)^n u_1 \) finishes off the proof i.e. \( u_{n+1} = (1 + \Delta)^n u_1 = u_1 + \left( \frac{n}{1} \right) \Delta u_1 + \left( \frac{n}{2} \right) \Delta^2 u_1 + \ldots + \left( \frac{n}{r} \right) \Delta^r u_1 + \ldots + \Delta^0 u_1 \)

(vii) First suppose that \( \Delta u_1 = \Delta u_2 = \ldots = \Delta u_{n-1} = 0 \). Then the following string of equalities hold:

\[
u_2 - u_1 = 0 \Rightarrow u_2 = u_1
\]

\[
u_3 - u_2 = 0 \Rightarrow u_3 = u_2 = u_1
\]

\[
u_4 - u_3 = 0 \Rightarrow u_4 = u_3 = u_2 = u_1
\]

\[
\ldots
\]

\[
u_n - u_{n-1} = 0 \Rightarrow u_n = u_{n-1} = \ldots = u_3 = u_2 = u_1
\]

Thus all the \( u_k \) are equal.

Now the reverse implication is where all the \( u_k \) are equal in which case it is clear that \( \Delta u_1 = \Delta u_2 = \ldots = \Delta u_{n-1} = 0 \). Thus the \( u_k \) are independent of \( n \).

(viii)

\[
\Delta S_n = S_{n+1} - S_n = u_{n+1}
\]

\[
= (1 + \Delta)^n u_1
\]

\[
= (1 + \Delta)^n \Delta v_1
\]

\[
= \Delta (1 + \Delta)^n v_1 \quad \text{using the properties of} \ \Delta \ \text{in (v) above}
\]

The critical step here is to note that \( u_{n+1} = (1 + \Delta)^n u_1 \). This follows from Newton's difference formula proved above.

From the above \( \Delta S_n - (1 + \Delta)^n v_1 = 0 \).

This is of the form \( \Delta (w_k - z_k) = \Delta x_k = 0 \) and (v) above indicates that differences are independent of \( n \). In other words \( S_n - (1 + \Delta)^n v_1 \) is independent of \( n \). In particular we can say that \( S_n - (1 + \Delta)^n v_1 = S_1 - (1 + \Delta) v_1 = u_1 - v_1 - u_1 = - v_1 \) (noting that it was assumed that \( \Delta v_0 = u_1 \) for all \( n \) so \( \Delta v_1 = u_1 \)).

Thus \( S_n = (1 + \Delta)^n v_1 = - v_1 \)

Hence \( S_n = (1 + \Delta)^n v_1 - v_1 = \{(1 + \Delta)^n - 1\} v_1 \)
45. An application of Fermat numbers

A Fermat number is defined for \( n = 0, 1, 2, \ldots \) as \( F_n = 2^{2^n} + 1 \). The first few Fermat numbers are as follows:

- \( F_0 = 3 \)
- \( F_1 = 5 \)
- \( F_2 = 17 \)
- \( F_3 = 257 \)
- \( F_4 = 65537 \)
- \( F_5 = 641 \cdot 6700417 \)

(i) Show using mathematical induction that \( \prod_{k=0}^{n-1} F_k = F_n - 2 \) for \( n \geq 1 \).

(ii) Having done (i) show that any two Fermat numbers must be relatively prime (i.e., their greatest common divisor (gcd) must be 1). Can you see how this fact might be used to prove there is an infinity of primes?

**Solutions**

(i) The formula is true for \( n = 1 \) since \( F_0 = 3 \) and \( F_1 - 2 = 5 - 2 = 3 \). Suppose that the formula is true for any \( n \) i.e., \( \prod_{k=0}^{n-1} F_k = F_n - 2 \).

We see that \( \prod_{k=0}^{n} F_k = (\prod_{k=0}^{n-1} F_k) F_n = (F_n - 2)F_n \) using the induction hypothesis.

\[
= (2^{2^n} - 1)(2^{2^n} + 1)
\]

\[
= 2^{2^{n+1}} - 1
\]

\[
= F_{n+1} - 2
\]

and so the proposition is true for \( n+1 \) and hence is established by induction.

(ii) Suppose \( m \) is a divisor of \( F_k \) and \( F_n \), say, where \( k < n \), then \( m \) must divide 2 since \( F_n = F_0 \cdot F_1 \cdot \ldots \cdot F_{n-1} + 2 \). This means that \( m = 1 \) or 2 but \( m = 2 \) is impossible since all Fermat numbers are obviously odd. Since there is an infinity of Fermat numbers (because there is an infinity of integers 1) and any two Fermat numbers are relatively prime there must be an infinity of primes.

46. A quick and useful result for polynomials

Consider \( p_n(x) = x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_0 \). If the roots (perhaps complex) are \( \lambda_1, \ldots, \lambda_n \), then \( p_n(x) \) can be written as the following product of factors:

\[ p_n(x) = \prod_{i=1}^{n} (x - \lambda_i) \]

**Problem**

Show that \( p_n(x) = \prod_{i=1}^{n} (x - \lambda_i) = x^n - \sum_{i} \lambda_i x^n - \sum_{i<j} \lambda_i \lambda_j x^{n-2} - \sum_{i<j<k} \lambda_i \lambda_j \lambda_k x^{n-3} + \ldots + (-1)^n \lambda_1 \lambda_2 \ldots \lambda_n \)

In the theory of symmetric functions the coefficients of \( x \) are usually denoted by:

\[ \sigma_1 = \lambda_1 + \lambda_2 + \ldots + \lambda_n \]

\[ \sigma_2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \ldots + \lambda_{n-1} \lambda_n \]

\[ \quad \ldots \]

\[ \sigma_n = \lambda_1 \lambda_2 \ldots \lambda_n \]

**Solution**

The proof is an inductive one. The proposition is uninformatively true for \( n = 1 \) so it is better to look at \( n = 2 \) as that gives a greater insight into how the proof unfolds.

\[
p_2(x) = \prod_{i=1}^{2} (x - \lambda_i) = x^2 - (\lambda_1 + \lambda_2) x + \lambda_1 \lambda_2 = x^2 - \sum_{i} \lambda_i x + (-1)^2 \lambda_1 \lambda_2 \]

so the proposition is true for \( n = 2 \). Note that from combinatorial considerations there are \( \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \) = 2 ways of choosing one \( \lambda \) at a time and \( \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \) = 1 way of choosing two \( \lambda \)’s at a time.

Now suppose that the proposition holds for any \( n \) and consider 

\[
p_{n+1}(x) = \prod_{i=1}^{n+1} (x - \lambda_i) = \prod_{i=1}^{n} (x - \lambda_i) (x - \lambda_{n+1})
\]

\[
= x^{n+1} - \sum_{i} \lambda_i x^n - \sum_{i<j} \lambda_i \lambda_j x^{n-2} - \sum_{i<j<k} \lambda_i \lambda_j \lambda_k x^{n-3} + \ldots + (-1)^n \lambda_1 \lambda_2 \ldots \lambda_n (x - \lambda_{n+1}) \]

\[
= \sum_{i} \lambda_i x^n - \sum_{i<j} \lambda_i \lambda_j x^{n-2} - \sum_{i<j<k} \lambda_i \lambda_j \lambda_k x^{n-3} + \ldots + (-1)^n \lambda_1 \lambda_2 \ldots \lambda_n x - \lambda_{n+1} x^n + \sum_{i} \lambda_i \lambda_{n+1} x^{n-1} - \sum_{i<j} \lambda_i \lambda_j \lambda_{n+1} x^{n-2} + \ldots + (-1)^n \lambda_1 \lambda_2 \ldots \lambda_n \lambda_{n+1}
\]

\[
= \sum_{i=1}^{n+1} \lambda_i x^n + \sum_{i<j} \lambda_i \lambda_j x^{n-2} - \sum_{i<j<k} \lambda_i \lambda_j \lambda_k x^{n-3} + \ldots + (-1)^{n+1} \lambda_1 \lambda_2 \ldots \lambda_n \lambda_{n+1}
\]

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Hence the proposition is true for \( n + 1 \).

47. An application of the Cauchy-Schwarz inequality to intervals for real roots of polynomials

Recall that the Cauchy-Schwarz inequality is stated as follows: for real \( a_i \) and \( b_j \) such that \( \sum a_i b_j \leq \left( \sum a_i^2 \right)^{1/2} \left( \sum b_j^2 \right)^{1/2} \).

Suppose all roots of the polynomial \( x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_0 \) are real. Show that the roots are contained in the interval with the endpoints

\[
\frac{-a_{n-1}}{n} \pm \frac{1}{n} \sqrt{a_{n-1}^2 - \frac{2n}{n-1} a_{n-2}}
\]

Hint: let one of the roots be \( \lambda \) (\( \lambda = \lambda_0 \) say) and express the polynomial in the form \( (x - \lambda)(x - \lambda_1)\ldots(x - \lambda_{n-1}) \). Then compare coefficients of \( a_{n-1} \) and \( a_{n-2} \). Once you have done that think about how the Cauchy-Schwarz inequality might be used (see Problem 46).

Solution

We know from problem 46 that the coefficient of \( x^{n-1} \) is \(-\sigma_1 = - (\lambda + \sum_{i=1}^{n-1} \lambda_i) \) or \( a_{n-1} = -(\lambda + \sum_{i=1}^{n-1} \lambda_i) \).

The coefficient of \( x^{n-2} \) is \( a_{n-2} = \sigma_2 = \lambda \sum_{i=1}^{n-1} \lambda_i \lambda_j \) (where it is to be noted that \( i \) starts at 1)

From these two pieces of information we have that:

\[
a_{n-2}^2 - 2a_{n-1} \lambda - \lambda^2 = \left( \lambda + \sum_{i=1}^{n-1} \lambda_i \right)^2 - 2 \lambda(\sum_{i=1}^{n-1} \lambda_i) - \lambda^2
\]

\[
= \lambda^2 + 2\lambda \sum_{i=1}^{n-1} \lambda_i \lambda_j - 2\lambda \sum_{i=1}^{n-1} \lambda_i - \lambda^2
\]

\[
= \left( \sum_{i=1}^{n-1} \lambda_i \right)^2 - 2 \sum_{i=1}^{n-1} \lambda_i \lambda_j
\]

\[
= \sum_{i=1}^{n-1} \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j - 2 \sum_{i=1}^{n-1} \lambda_i \lambda_j
\]

This is where we can use the Cauchy-Schwarz inequality. Consider the following:

\[
(a_{n-1} + \lambda)^2 = \left( \sum_{i=1}^{n-1} \lambda_i \right)^2
\]

\[
(n-1) \sum_{i=1}^{n-1} \lambda_i^2 \quad \text{[Because you can apply Cauchy-Schwarz to } \lambda_1, \ldots, \lambda_{n-1} \text{ and } 1, \ldots, 1 \text{ (there being } (n-1) 1's \text{ here) to get } \left( \sum_{i=1}^{n-1} \lambda_i \right)^2 \sum_{i=1}^{n-1} \lambda_i^2 = (n-1) \sum_{i=1}^{n-1} \lambda_i^2 \]}

\[
= (n-1) (a_{n-1}^2 - 2a_{n-2} - \lambda^2)
\]

ie \( a_{n-1}^2 + 2\lambda a_{n-1} + \lambda^2 \) \( (n-1) \) \( a_{n-1}^2 - 2a_{n-2} - \lambda^2 \)

\[
n\lambda^2 + 2n\lambda a_{n-1} - (n-2) a_{n-1}^2 + 2(n-1) a_{n-2} = 0
\]

Therefore, \( \lambda^2 + \frac{2n\lambda}{n-2} a_{n-1} - \frac{2(n-1)}{n-2} a_{n-2} = 0 \) and thus \( \lambda \) \( \text{and hence all the } \lambda_i \) \( \text{because there was nothing special about our choice for } \lambda \) \( \text{lie between the two roots of this quadratic function. So the roots of that function are the relevant bounds. The roots of that} \)

\[
\text{function are } \frac{-a_{n-1}}{n} \pm \sqrt{\frac{\left( \frac{2(n-1)}{n} a_{n-2} \right)^2 - 4 \left( \frac{2(n-2)}{n} a_{n-1} - \frac{2(n-1)}{n} a_{n-2} \right)}{2}} = \frac{-a_{n-1}}{n} \pm \sqrt{\frac{\left( \frac{2(n-1)}{n} a_{n-2} \right)^2 - 4 \left( \frac{2(n-2)}{n} a_{n-1} - \frac{2(n-1)}{n} a_{n-2} \right)}{2}}
\]
48. Using induction to solve functional equations

Cauchy did fundamental analytical work on functional equations hence there is a basic theorem named after him. Here it is. Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function satisfying:

\[
f(x + y) = f(x) + f(y)
\]

for all real \( x \) and \( y \).

From this meagre information one can show that there exists a real number a such that \( f(x) = ax \) for all \( x \in \mathbb{R} \).

Welcome to the theory of functional equations!

One can use induction to show that \( f(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} f(x_i) \) for all \( x_i \in \mathbb{R} \). For \( n=2 \) the formula is true since it is just a restatement of the definition. Assume the formula is true for any \( n \) and consider \( f(\sum_{i=1}^{n+1} x_i) = f(\sum_{i=1}^{n} x_i + x_{n+1}) = \sum_{i=1}^{n} f(x_i) + f(x_{n+1}) \) using the induction hypothesis and the definition of \( f(x+y) \).

\[
= \sum_{i=1}^{n+1} f(x_i)
\]

Hence the formula is established for \( n+1 \). When the \( x_i = x \) for all \( i \) we have that \( f(nx) = n f(x) \) for all positive integers and all real \( x \). This may seem a pretty trivial result but the plan here is to use this to establish that the functional equation holds for rational scaling and then make the leap to scaling by a real number. It is very common to use induction to establish some basic structure of the functional equation that can be manipulated to get to where you want. Here we want to establish that \( f(x) = ax \) for some real \( a \).

We can make the leap to rational scaling by letting \( x = \frac{m}{n} \) where \( m \) and \( n \) are positive integers. Because \( nx = mt \) we have that \( f(nx) = f(mt) \) and using the result we just established by induction we see that:

\[
n f(x) = m f(t)
\]

\[
n f(\frac{m}{n} t) = m f(t)
\]

So we have that \( f(\frac{m}{n} t) = \frac{m}{n} f(t) \) for all \( t \in \mathbb{R} \). In other words since \( \frac{m}{n} \) is rational we have established that \( f(qt) = q f(t) \) for all \( t \in \mathbb{R} \) and positive rational \( q \). We now need to pick up \( q < 0 \). This isn’t too hard because \( f(x + 0) = f(x) + f(0) = f(x) + f(0) \) so \( f(0) = 0 \). This means that \( f(0) = 0 \). Thus \( q = 0 \) is picked up. To pick up \( q < 0 \) we start with:

\[
0 = f(0)
\]

\[
= f(\ q + (-q))
\]

\[
= f(q) + f(-q)
\]

Hence \( f(-q) = -f(q) \). If \( q < 0 \) then \( f(-(-qt)) = f((-qt)) = -f(q) \). So we have shown that \( f(qt) = q f(t) \) for all rational \( q \) and real \( t \).

If we let \( t = 1 \) we get \( f(q.1) = q f(1) \) ie \( f(q) = q f(1) \). Let \( f(1) = a \) (a real number) we have that \( f(q) = q a \) for all rational \( q \).

The end game is to establish that we can say that \( f(x) = ax \) for \( x \) real by leaping from the rationals to reals. This is where continuity comes in. I won’t provide the full proof but the gist is that the rationals can approximate a real to an arbitrary degree of accuracy.

**Problem**

With this theoretical background let’s do a “meaty” problem which involves both induction and the ideas discussed above.

Suppose you are told there is some function \( f(x) \) which is strictly positive with the following properties for all real \( x \):

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(a) \( f(2x) = [f(x)]^2 \)
(b) \( f(-x) = [f(x)]^{-1} \)
(c) \( f(x) = 1 + x \)

(1) Make an inspired guess what a candidate for this function would be. It has figured more than once in this problem set.

(2) Prove by induction that \( f(x) = [f\left(\frac{x}{2}\right)]^n \) for integers \( n \geq 1 \).

(3) Use (2) to show that \( \left(1 + \frac{x}{2^n}\right)^{2^n} f(x) \) for integers \( n \geq 1 \). Have a close look at \( \left(1 + \frac{x}{2^n}\right)^{2^n} \) - does it remind you of anything?

(4) Use similar reasoning to show that \( f(-x) = \left(1 - \frac{x}{2^n}\right)^{2^n} \) for integers \( n \geq 1 \).

(5) "Sandwich" \( f(x) \) between \( \left(1 + \frac{x}{2^n}\right)^{2^n} \) and an upper bound. To get this upper bound show that \( f(x) = [f(-x)]^{-1} \) and derive an upper bound of the form \( \left(1 + \frac{x}{2^n}\right)^{2^n} g(x,n) \). You should aim for this: \( \left(1 + \frac{x}{2^n}\right)^{2^n} f(x) \leq \left(1 + \frac{x}{2^n}\right)^{2^n} g(x,n) \) where \( g(x,n) \) is some expression involving \( a \) and \( n \). Now if you guessed the form of \( f(x) \) correctly (and the 3 conditions make it pretty clear what is should be) use a simple limit argument to conclude that your guess is the unique solution to this functional equation problem.

**Solution**

(1) Your guess should be \( f(x) = e^x \). Checking it:

(a) \( f(2x) = (e^x)^2 = e^{2x} \)
(b) \( f(-x) = e^{-x} = \frac{1}{e^x} = (e^x)^{-1} \)
(c) \( f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \) \((1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots) = 1 + x \) (we are told \( f(x) \) is strictly positive and we know that \( e^x > 0 \) for all \( x \))

(2) \( f(x) = [f\left(\frac{x}{2}\right)]^n \) is true for \( n = 1 \) for the following reason:

\[ f(x) = f(2 \cdot \frac{x}{2}) = [f\left(\frac{x}{2}\right)]^2 \] using property (a). Now suppose that \( f(x) = [f\left(\frac{x}{2}\right)]^n \) is true for any \( n \geq 1 \). Then:

\[ f(x) = [f\left(\frac{x}{2}\right)]^{2^n} = [f(2 \cdot \frac{x}{2^{n+1}})]^{2^n} = [f\left(\frac{x}{2^{n+1}}\right)]^{2^n} = [f\left(\frac{x}{2^{n+1}}\right)]^{2^{n+1}} \] using property (a) and the induction hypothesis. Hence the formula is true for \( n+1 \).

(3) \( f(x) = 1 + \frac{x}{2} \) by property (c) of the definition of \( f(x) \). Then using (2), \( f(x) = [f\left(\frac{x}{2}\right)]^{2^n} \left(1 + \frac{x}{2^n}\right)^{2^n} \). Note that if \( 1 + \frac{x}{2^n} > 0 \) then any power of it is \( > 0 \). If \( 1 + \frac{x}{2^n} < 0 \), then any even power of it will be \( > 0 \). Either way the direction of the inequality is as shown.

(4) \( f\left(\frac{x}{2^n}\right) = 1 - \frac{x}{2^n} \) by property (c) of the definition of \( f(x) \) and noting that the property holds for all real \( x \). Then by the same reasoning in (3) \( f(x) = [f\left(\frac{x}{2^n}\right)]^{2^n} \left(1 - \frac{x}{2^n}\right)^{2^n} \).

(5) This is where the real action begins. We have that \( \left(1 + \frac{x}{2^n}\right)^{2^n} f(x) \) and \( f(x) - \left(1 - \frac{x}{2^n}\right)^{2^n} \) and we want to "sandwich" \( f(x) \) between the same number so that we can assert that \( f(x) \) is unique. This technique is frequently used in analysis when studying convergence of sequences and series. First note that \( f(-x) = \frac{1}{f(x)} \left(1 - \frac{x}{2^n}\right)^{2^n} \) so that \( f(x) - \left(1 - \frac{x}{2^n}\right)^{2^n} \).

Note again that the direction of the inequality is correct because we are dealing with even powers of \( \left(1 - \frac{x}{2^n}\right) \). So we have \( \left(1 + \frac{x}{2^n}\right)^{2^n} f(x) \) \( \left(1 + \frac{x}{2^n}\right)^{-2^n} \). Now \( \left(1 + \frac{x}{2^n}\right)^{-2^n} = \left(1 - \frac{x}{2^n}\right) = \frac{1}{\left(1 - \frac{x}{2^n}\right)^{2^n}} \) \( \frac{1}{\left(1 - \frac{x}{2^n}\right)^{2^n}} \).

Note that \( 4^n = (2^n)^{2^n} \). So we have \( \left(1 + \frac{x}{2^n}\right)^{2^n} f(x) \) \( \left(1 - \frac{x}{2^n}\right)^{-2^n} \left(1 + \frac{x}{2^n}\right)^{2^n} \). Now when we fix \( x \) and let \( n \to \infty \), \( \left(1 + \frac{x}{2^n}\right)^{2^n} \to e^x \) and \( \left(1 - \frac{x}{2^n}\right)^{-2^n} \to 1 \) Here is a visual demonstration:
He then defined a function \( f(z) \) = b.

In order to show that \( g(z) = az \) for some a, Cauchy had to establish that both \( f \) and \( \log f \) were continuous. Having done all of the above he wanted to show by standard analytical techniques that a remainder approaches zero as \( n \to \infty \) that principle applied \( 2^n \) times gives the result.

So what we have is this: \( e^x \) \( f(x) \) = \( e^x \). This means our guessed function is the unique solution.

### 49. Filling in the gaps for Newton

The proof of the binomial theorem in its most general form is actually reasonably intricate and it is little wonder that Newton’s proof falls short of today’s standards of analytical rigour. You can read for yourself how Newton guessed the result by going to this link: http://www.macalester.edu/~aratra/edition2/chapter2/chapt5d.pdf.

This is not a problem about induction, rather it is a detailed piece of analysis that underpins the confidence you can have in using the binomial theorem. There is a lot of basic "entry level" analysis in this series of problems.

What I have done below is break the proof down into a series of steps which build on one another. Some analysis is required but I have kept the knowledge to a minimum. Those who know more can see how one might compress one or two of the steps but for the most part you have to show by standard analytical techniques that a remainder approaches zero as \( n \to \infty \). Some of the sub-steps could be treated as self-contained problems simply because they involve quite fundamental limiting arguments.

Cauchy showed that \((1 + x)^n\) does converge to a finite real number for all real \( z \) if \( |x| < 1 \).

He then defined a function \( f(x) = 1 + zx + \left( \begin{array}{l} \frac{z}{2} \\ 2 \end{array} \right) x^2 + \left( \begin{array}{l} \frac{z}{3} \\ 3 \end{array} \right) x^3 + ... \) and showed by combinatorial arguments and rules for multiplying two such infinite sums that

\[
 f(z + w) = f(z) f(w) \quad \text{for all real } z \text{ and } w.
\]

Cauchy then wanted to show that by taking logarithms he could get something like \( g(z + w) = g(z) + g(w) \) where \( g = \log f \).

In order to show that \( g(z) = az \) for some a Cauchy had to establish that both \( f \) and \( \log f \) were continuous. Having done all of that he deduced that \( f(z) = b^z \) for some b. The final step was to note that \( b = 1 + x \) which follows by noting that \( f(1) = 1 + x + \left( \frac{1}{2} \right) x^2 + ... = 1 + x \) since \( \left( \frac{1}{j} \right) = 0 \) for \( j > 1 \).

One can use Taylor’s Theorem to show that

\[
 f(x) = (1 + x)^n = 1 + mx + \left( \frac{m}{2} \right) x^2 + \left( \frac{m}{3} \right) x^3 + ...
\]

does converge for any m rational and \( |x| < 1 \). This problem sets out a series of steps to establish this.

There is a lot in this problem but I have kept the analytical parts as basic as possible. I have stated the bits of analysis that you need to make the relevant deductions. You do need to know that differentiability implies continuity and I assume you have probably proved this. I also assume you have used the Fundamental Theorem of Calculus.

**Problem**

1. Suppose \( f(x) \) has continuous derivatives up to order \( n \) and let:

\[
 F_n(x) = f(b) - f(x) - (b - x) f'(x) - ... - \frac{(b-x)^n}{n!} f^{(n)}(x)
\]
Show that $F_n'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x)$

(2) Hence show that $F_n(a) = F_n(b) - \int_a^b F_n'(x) \, dx = \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x) \, dx$

(3) By letting $b = a + h$ and an appropriate substitution in the integral in (2) show that:

$$f(a + h) = f(a) + h f'(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

where $R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(a + th) \, dt$

(4) If $f(x) \neq 0$ for $x \in [a, b]$, then show that if $M f(x) N$ for $x \in [a, b]$ implies that $M (b - a) \int_a^b f(x) \, dx N (b - a)$ (You can assume this but you should be able to see why it is true - draw a diagram and it hits you in the face). Using these building blocks (and noting that $f$ is continuous since it is differentiable) show that $\exists \xi \in (a, b)$ such that $\int_a^b f(x) \, dx = (b - a) f(\xi)$

(5) If $M f(x) N$ for $x \in [a, b]$ and $\psi(x) \neq 0$ then show that $M \int_a^b \psi(x) \, dx N \int_a^b \psi(x) \, dx$. Using similar reasoning to (4), show that $\exists \xi \in (a, b)$ such that $\int_a^b \psi(x) \, dx = f(\xi) \int_a^b \psi(x) \, dx$

(6) Using (5) show that when $0 < \theta < 1$ and $1 \leq p, n$, $R_n = \frac{(1-\theta)^n \cdot f^{(n)}(a + \theta h) \cdot \theta^p}{p!(n-1)!} \cdot (a + \theta h) \int_0^1 (1-t)^{n-1} \, dt$

As a first step justify this: $\int_0^1 (1-t)^{n-1} \, dt = \frac{(1 - \theta)^{n-1} \cdot f^{(n)}(a + \theta h) \cdot \theta^p}{p!(n-1)!}$

Note that from (3) $R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} \, dt$

You can assume that a function $g(x)$ which is continuous on a closed, bounded interval such as $[a, b]$ is bounded (and in fact attains its maximum on the interval). In other words $\exists K > 0$, such that $|g(x)| \leq K$ for all $x \in [a, b]$

If $p = n$ the remainder $R_n$ is called Lagrange’s form while if $p = 1$ it is called Cauchy’s form, namely:

$$R_n = \frac{(1-\theta)^n \cdot f^{(n)}(a + \theta h) \cdot \theta^p}{p!(n-1)!}$$

(7) Now consider $f(x) = (1 + x)^m$ where $m$ is not a positive integer but is some rational number. Show that the Cauchy form of the remainder in the Taylor series expansion is $R_n = \frac{m(m-1) \cdots (m-n+1)(1-x)^{n-1} x^n}{(n-1)!}$. Note here that for rational $m$, $m(m-1) \cdots (m+n-1)$ still makes sense.

(8) Show that $1 + \theta x < 1 \text{ as long as } -1 < x < 1$ and $1 - |x|^{m-1} < (1 + \theta x)^{m-1}$ if $m > 1$ and $(1 + \theta x)^{m-1} < (1 - |x|)^{m-1}$ if $m < 1$

(9) Show that $1 \leq (1 + \theta x)^{m-1} < (1 + |x|)^{m-1}$ if $m > 1$ and $1 \leq (1 + \theta x)^{m-1} < (1 - |x|)^{m-1}$ if $m < 1$

(10) Show that $|R_n| < |M| (1 \pm |x|)^{m-1} \left( \frac{m-1}{m-n+1} \right) \cdot |x|^n$. We want to show that this approaches 0 as $n \to \infty$ and the next steps are directed at establishing that. In other words we have to show that $R_n \to 0$.

(11) Show that and if $\psi(n) > 0$ for all $n$ and if $|\psi(n+1)| < K |\psi(n)|$ when $n N$ and $0 < K < 1$, then $\lim_{n \to \infty} \psi(n) = 0$

(12) By using (11), show that if $\lim_{n \to \infty} \frac{\psi(n+1)}{\psi(n)} = \lambda$ where $-1 < \lambda < 1$, then $\lim_{n \to \infty} \psi(n) = 0$

(13) Use (12) to show that $R_n \to 0$. Having done all of that you will have proved that the binomial expansion holds for any rational $m$ and $-1 < x < 1$. Think how irrational values of $m$ might be approached.

Solutions

(1) Since $F_n(x) = f(b) - f(x) - (b-x)f'(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) = f(b) - \sum_{k=0}^{n-1} \frac{(b-x)^k}{k!} f^{(k)}(x)$

$F_n(x) = \sum_{k=0}^{n-1} \frac{-(b-x)^k}{k!} f^{(k)}(x) + \sum_{k=0}^{n-1} \frac{(b-x)^k}{k!} f^{(k+1)}(x)$ (just the derivative of a product with some sigma notation wrapped around it - note that $f(b)$ is simply a constant)
\[ f(a + h) = f(a) + h \quad f'(a) + \cdots + \frac{h^{n}}{n!} f^{(n)}(a) + R_{n} \]
Another way of obtaining the result is to note that since \( \int_a^b (f(x) - M) \, dx \) 0 and \( \int_a^b (N - f(x)) \, dx \) 0 (because \( M \leq f(x) \leq N \)) for x \( \epsilon \) [a, b], \( \int_a^b f(x) \, dx \) = \( M(b-a) \) and \( \int_a^b f(x) \, dx \) = \( N(b-a) \).

You have to show that \( \exists \xi \in (a, b) \) such that \( \int_a^b f(x) \, dx = (b-a) f(\xi) \). Since \( M \leq f(x) \leq N \) for x \( \epsilon \) [a, b] the integral is equal to \( \eta(b-a) \) where \( M \leq \eta \leq N \). This is a pretty subtle but important observation. All it is saying is that the area is a length \( (b-a) \) times some other real number \( \eta \) (a length). This must be true. However \( f \) is continuous on \([a, b]\) so there must be an \( \xi \) between a and b such that \( f(\xi) = \eta \) (a version of the Intermediate Value Property). Hence \( \int_a^b f(x) \, dx = (b-a) f(\xi) \) for this \( \xi \). In Analysis courses you would prove that if \( f \) is integrable in the Riemann sense, \( \int_a^b f(x) \, dx \) is continuous and you can deduce the result using the continuity of the integral rather than just the function itself. You can see now why the point is a subtle one but it avoids the overhead of establishing things that require much more knowledge.

You have to show that \( \exists \xi \in (a, b) \) such that \( \int_a^b f(x) \, dx = (b-a) f(\xi) \). Since \( M \leq f(x) \leq N \) for x \( \epsilon \) [a, b] it follows that \( \int_a^b f(x) \, dx - M \) \( f(\xi) \, dx \) 0 and \( \int_a^b (N - f(x)) \psi(x) \, dx \) 0. Now the value of the integral \( \int_a^b f(x) \psi(x) \, dx \) is simply some number \( \lambda \) times \( \int_a^b f(x) \, dx \) (the latter integral gives rise to a number 0). But \( M \leq \lambda \leq N \) and since \( M \leq f(x) \leq N \) for x \( \epsilon \) [a, b] and \( f \) is continuous, there is an \( \xi \) in (a,b) such that \( \lambda = f(\xi) \). Hence, for this \( \xi \), \( \int_a^b f(x) \psi(x) \, dx = f(\xi) \int_a^b \psi(x) \, dx \)

(6) If we let \( p \) be an integer 1 \( \leq p \leq n \) then we can write \( R_n \) in the following way: \( R_n = \int_{(a+1)^{n-1}}^{(b+1)^{n-1}} (1 - t)^{n-1} f^{(n)}(a + th) \, dt = \int_{(a+1)^{n-1}}^{(b+1)^{n-1}} (1 - t)^{n-1} f^{(n)}(a + th) \, dt \). There is nothing dodgy about this because 1 \( \leq p \leq n \) so you don't get a negative exponent which might cause problems with the function "blowing up" on [0,1]. The claim is that \( R_n \) = \( \frac{(1 - 0)^{n-1} f^{(n)}(a + th) b^p}{p(n-1)!} \) rather than being less than or equal to something. This suggests we might be able to use (5) to some advantage. Basically we have an integral of this form:
\[ \int_0^1 (1-t)^{-p} \, dt. \]

Looking at \[ \int_0^1 (1-t)^{-p} \, dt \] the idea is to get something that satisfies the hypotheses of (5). Thus we need \( g(x) \) and \( \phi(x) \) such that

\[ \int_0^1 g(x) \, dx = T \] and \( \phi(x) = 0 \) for \( x \in [a,b] \) then we will have \( \int_0^1 \phi(x) \, dx \) and finally that \( \Delta \) between a and b such that

\[ g(x) \phi(x) \, dx = \frac{1}{p(n-1)!} \int_0^1 g(x) \, dx. \]

If we take \( g(t) = (a + \theta) (1-t)^{-p} \) and \( \phi(t) = (1-t)^{-p-1} \) we can assert that \( g \) is bounded on \([0,1]\) and that \( \phi(t) \rightarrow 0 \) on \([0,1]\). The reason \( g \) is bounded on \([0,1]\) is that \( (1-t)^{-p} \) is clearly bounded (it doesn’t “blow up” because \( n - p > 0 \)) and we assumed that \( f \) has continuous derivatives up to order \( n \). This means that \( f^{(n)}(a + \theta) \) is continuous on \([0,1]\) and hence must be bounded there. Hence \( \exists K > 0 \) such that \( f^{(n)}(a + \theta) \mid K \) for all \( t \in [0,1] \). The product of two bounded functions is itself bounded.

With this identification of \( g(t) \) and \( \phi(t) \) we have that \( \exists \theta \in (0,1) \) such that

\[ \int_0^1 (1-t)^{-p} \, dt = (1-\theta)^{-p} \int_0^1 (a + \theta) \, dt. \]

Using the substitution \( u = 1 - t \) in \( (1-t)^{-p} \) the integral becomes

\[ \int_0^1 w^{-1} \, dw = \int_0^1 w^{p-1} \, dw = \frac{1}{p}. \]

Putting this all together we get \( R_n = \frac{1}{p(n-1)!} \) as claimed

(7) As indicated above the Cauchy form (note that \( p = 1 \) for the Cauchy form) of \( R_n \) is \( \frac{1}{n!} \). It is easy to differentiate \( f(x) = (1 + x)^m \) \( n \) times : 

\[ f^{(n)}(x) = m(m-1)\cdots(m-n+1) (1 + x)^{m-n}. \]

If you have got this far it is too insulting to ask you to do an inductive proof of this!

Thus \( R_n = \frac{\sigma(m-n+1)\cdots(m-n+1)(1-\theta)^{m-n}}{n!} \) (here \( x \) takes the place of \( a + \theta \) and \( a = 1 \))

(8) If \( -1 < x < 1 \) then because \( 0 < \theta < 1, \theta - \theta x < \theta \) and hence \( 0 < 1 - \theta + \theta x \). Thus \( \frac{1-\theta}{1+\theta x} < 1 \)

(9) If \( -1 < x < 1 \) then because \( 0 < \theta < 1, \theta x > |x| \). So for \( m > 1 \), \( (1 + \theta x)^{m-1} < (1 + |x|)^{m-1} \).

Now \( \theta x > |x| \) is clearly true for \( x > 0 \). When \( x < 0 \), one has \( -\theta |x| > |x| \) which is also true since \( 0 < \theta < 1 \). Thus \( 1 + \theta x > 1 - |x| \) and so when the exponent is negative \( m < 1 \), it follows that \( (1 + \theta x)^{m-1} < (1 - |x|)^{m-1} \). Thus \( (1 + \theta x)^{m-1} < (1 + |x|)^{m-1} \).

(10) From (7), (8) & (9) we have that

\[ |R_n| = \frac{\sigma(m-n+1)\cdots(m-n+1)(1-\theta)^{m-n}}{n!} \]

\[ = \frac{m(m-1)\cdots(m-n+1)(1-\theta)^{m-n}}{n!} \]

\[ = |m||m-1||m-2| \cdots |m-n+1| \frac{(1-\theta)^{m-n}}{n!} \]

\[ < |m||m-1||m-2| \cdots |m-n+1| \frac{(1-\theta)^{m-n}}{n!} \]

\[ < |m||m-1||m-2| \cdots |m-n+1| \frac{(1-\theta)^{m-n}}{n!} \]

\[ = \sigma_n. \]

So if we can show that \( \sigma_n \rightarrow 0 \) as \( n \rightarrow \infty \), it will follow that \( R_n \rightarrow 0 \).

(11) We have to show that if \( \psi(n) > 0 \) for all \( n \) and if \( |\psi(n+1)| < K |\psi(n)| \) when \( n \) \( N \) and \( 0 < K < 1 \), then \( \lim_{n \rightarrow \infty} \psi(n) = 0 \). It is a straightforward inductive gambit to show that \( |\psi(n+1)| < K |\psi(n)| < K^2 |\psi(n-1)| < \cdots < K^n |\psi(1)| \). To say that \( \lim_{n \rightarrow \infty} \psi(n) = 0 \) is to say that we can find a positive \( \epsilon \) no matter how small such that \( |\psi(n)| < \epsilon \) for all sufficiently large \( n \). This is exactly what we have established since if we choose \( \epsilon \) we require \( K^n |\psi(1)| < \epsilon \) and we can certainly make \( n \) sufficiently large to achieve this (note here that \( 0 < K < 1 \) and \( |\psi(1)| \) is just some positive number).

(12) By using (11), we have to show that if \( \lim_{n \rightarrow \infty} \frac{\phi(n+1)}{\phi(n)} = \lambda \) where \( -1 < \lambda < 1 \), then \( \lim_{n \rightarrow \infty} \phi(n) = 0 \).

\[ |\psi(n+1)| = \left| \frac{\psi(n+1)}{\psi(n)} \right| |\psi(n)| = \left| \frac{\psi(n+1)}{\psi(n)} \right| |\psi(n)| = \left| \frac{\psi(n+1)}{\psi(n)} - \lambda + \lambda \right| |\psi(n)| \]

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Now we can make 
\[
\left| \frac{\phi(n+1)}{\phi(n)} - \lambda \right| + |\lambda| \to \frac{\phi(n)}{|\lambda|} \text{ as small as we like since } \lim_{n \to \infty} \frac{\phi(n+1)}{\phi(n)} = \lambda \ . \text{ Recall that means that for any } \epsilon > 0 \text{ (matter how small) there is a } N \in \mathbb{Z}^+ \text{ such that }
\left| \frac{\phi(n+1)}{\phi(n)} - \lambda \right| < \epsilon \text{ for all } n > N . \text{ Thus:}
\[
|\phi(n+1)| - \lambda | + |\lambda| \begin{cases} \phi(n) \text{ if } |\epsilon + |\lambda|| |\phi(n)| = K |\phi(n)| \text{ where } K = \epsilon + |\lambda| \end{cases} \ . \text{ Now because } |\lambda| < 1 \text{ we can choose } \epsilon \text{ so small so that } \epsilon + |\lambda| < 1 \text{ ie } 0 < K < 1 . \text{ Thus the solution to (11) allows to conclude that } \lim_{n \to \infty} \phi(n) = 0 .
\]

(13) Recall from (10) that \[ |R_n| = \left| \frac{m(n+1)\ldots(n-m+1)(1-\theta)^m}{(n-m)!(1+\theta)^m} \right| < |m| \left( \frac{m-1}{n-1} \right) (1 + |x|)^{m-1} |x|^n = \sigma_n . \] Now consider \[ \lim_{n \to \infty} \frac{\sigma_{n+1}}{\sigma_n} = \left( \frac{n-1}{n} \right)^{x^{n+1}} \left( \frac{m-n}{n} \right)^{x^n} = \frac{m-n}{n} \to |x| < 1 \text{ since } -1 < x < 1 \text{ (note that } m \text{ is fixed number)} .
\]
This demonstrates that the remainder converges to zero and hence the binomial series converges for all rational \( m \) when \(-1 < x < 1\). When \( m \) is irrational we need to work with a different formulation, namely, \((1 + x)^m = e^{m \ln(1 + x)}\). The same reasoning used above can be replicated once it is noted that \[ \frac{d}{dx} (1 + x)^m = m (1 + x)^{m-1} . \]

50. The Poisson process

The Poisson process is a continuous time analogue of the discrete Bernoulli process. The Bernoulli process is usually characterized in terms of independent coin tosses where the probability of a head is fixed number \( p \) such that \( 0 < p < 1 \). Each trial produces a success (head) with probability \( p \) and a failure (tail) with probability \( 1 - p \) independently of what happens with the other trials. The independence assumption leads to a memorylessness or “fresh start” property. Thus for any given time \( t \), the sequence of random variables \( X_t \) (ie the future process) is also a Bernoulli process which is independent of the past process. To recapitulate some basic background, the Bernoulli process is binomial with parameters \( p \) and \( n \). Thus its probability “mass” function \( k \) successes in \( n \) independent trials is given by:

\[
p_s(k,n) = n \binom{n}{k} p^k (1 - p)^{n-k} \text{ for } k = 0, 1, 2, \ldots, n
\]

For the record the mean number of successes is \( E[S] = np \) while the variance is \( \text{var}[S] = np(1-p)\).

If one were to model something like traffic accidents in a city, we could make our observations in the fixed time interval of one minute, say, and we might treat a “success” as the observation of at least one accident in the period. The Bernoulli process does not keep track of the number of accidents in the period since the universe of outcomes is either a “success” (at least one accident) or a “failure” (no accidents). Because the Bernoulli process does not contemplate a natural way of dividing time into discrete bits there is a loss of information eg there were 20 accidents between 10.00 am and 10.01 am but only 1 between 10.02 am and 10.03 am, but both periods reflect a “success”.

The Poisson process reflects a continuous time approach to such problems. There are several ways of approaching the Poisson distribution. For instance it can be derived by limiting arguments based on occupancy problems. Thus Feller derives the probability \( p_m(r,n) \) of finding exactly \( m \) cells empty when a random distribution of \( r \) balls is made into \( n \) cells. This probability is:

\[
p_m(r,n) = \binom{n}{m} \sum_{j=0}^{m-n} (-1)^j \binom{m-j}{n} (1 - \frac{m+j}{n})^r
\]
It can then be shown that \( p_m(r,n) \leq e^{-\lambda} \frac{\lambda^m}{m!} \to 0 \) (where \( m \) is fixed and \( r \) and \( n \) tend to infinity).


There are other ways of approaching the development of the distribution and in this problem we look at two different approaches, one is a limiting approach which involves a basic inductive argument, while the other approach is based on developing an appropriate differential equation, the solution to which involves some inductive reasoning. The second approach opens up an important technique that is fundamental for any advanced study of probability theory.

(1) Deriving the Poisson process by using the binomial distribution and some inductive arguments

The starting point for this derivation is a Bernoulli process where \( n \) is large and \( p \) is small but \( \lambda = np \) is of moderate size. This is how Poisson derived the distribution that is named after him. We start at the bottom and work our way up. Thus if there are \( k = 0 \) successes the probability is \( b(0,n,p) = (1 - p)^n = \left( 1 - \frac{1}{n} \right)^{mn} \). Written this way this should look eerily familiar since \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e^t \) so for large \( n \) one would assert that \( b(0,n,p) = e^{-\lambda} \) which gives log \( b(0,n,p) = n \log \left( 1 - \frac{1}{n} \right) \) and using Taylor’s Theorem one then gets:
\[ \log b(0,n,p) = n \log \left( 1 - \lambda \cdot \frac{1}{n} \right) = -\lambda - \frac{\lambda^2}{2n} - \ldots \]. Hence for large \( n \), \( b(0,n,p) \sim e^{-\lambda} \) since the terms of order \( \frac{1}{n} \) and beyond are ignored in the approximation.

**Problem 1**

By analysing the ratio \( \frac{b(k,n,p)}{b(k-1,n,p)} \) for any fixed \( k \) and sufficiently large \( n \) show that \( b(1,n,p) \sim \lambda e^{-\lambda} \) and then show generally by induction that \( b(k,n,p) \sim \frac{\lambda^k}{k!} e^{-\lambda} \).

**Solution**

The starting point is the definition \( b(k, n, p) = \binom{n}{k} p^k q^{n-k} \).

\[
\frac{b(k,n,p)}{b(k-1,n,p)} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}}
\]

\[
= \frac{n!}{(n-k+1)! k!} \frac{(k-1)! (n-k+1)!}{n!} \frac{p}{q}
\]

\[
= \frac{(n-k+1)p}{kq}
\]

\[
= \frac{\lambda - (k-1)p}{kq} \quad \text{since } np = \lambda
\]

\[
= \frac{\lambda - (k-1)p}{k(1-p)} \quad \text{since } q = 1 - p
\]

\[
= \frac{\lambda}{k(1-p)} \left( \frac{1}{1-p} \right) \quad \text{since } p \text{ is assumed small so that } \frac{1}{1-p} \sim 1 \text{ and } \frac{p}{1-p} \sim 0
\]

Having established this we build up an inductive argument as follows:

Because \( \frac{b(k,n,p)}{b(k-1,n,p)} = \frac{\lambda}{k} \) it follows that \( b(1,n,p) \sim \lambda b(0,n,p) \quad \sim \lambda e^{-\lambda} \) (see above preamble to the problem).

and \( b(2,n,p) \sim \frac{\lambda}{2} b(1,n,p) \quad \sim \frac{\lambda^2}{2} e^{-\lambda} \). Thus a reasonable guess would be that \( b(k,n,p) \sim \frac{\lambda^k}{k!} e^{-\lambda} \).

To prove this by induction we note that we have established the base case of \( k = 1 \) above. Assume that our guess is true for any integral \( k > 1 \).

Then \( b(k+1,n,p) \sim \frac{\lambda}{k+1} b(k,n,p) \sim \frac{\lambda^{k+1}}{(k+1)!} e^{-\lambda} \) and so the guess is established by induction.

**Problem 2**

First, some background.

The theory of Poisson processes is usually taught by reference to some generic properties which can be set out as follows. The probability that there are exactly \( k \) arrivals during an interval (usually time) of length \( \tau \) is \( P(k, \tau) \). It is assumed that this probability is the same for all intervals of length \( \tau \). It is also assumed there is a positive parameter \( \lambda \) which is the arrival rate or intensity of the process.

More formally the process can be defined as follows:

\[(1) \text{ The probability } P(k, \tau) \text{ of } k \text{ arrivals is the same for all intervals of the same length } \tau. \text{ In essence this means that arrivals are } \text{"equally likely" at all times. In the Bernoulli context the analogue is that the probability of success is the same for all trials.} \]

\[(2) \text{ The number of arrivals during a particular interval is independent of the history of arrivals outside that period. The probability of } k \text{ arrivals during an interval of } T - t \text{ is } P(k, T - t) \text{ and even if we have some information about arrivals outside } [t, T] \text{ the conditional probability of } k \text{ arrivals during } [t, T] \text{ is the same as the unconditional probability } P(k, T - t). \]
(3) The probabilities \( P(n, t) \) satisfy:
\[
P(0, t) = 1 - \lambda t + o(t) \\
P(1, t) = \lambda t + o_2(t) \\
P(k, t) = o_k(t) \quad \text{for } k = 2, 3, \ldots
\]

In this context \( o(\tau) \) and \( o_k(\tau) \) are functions which have the properties that
\[
\lim_{\tau \to 0} \frac{o(\tau)}{\tau} = 0 \quad \text{and} \quad \lim_{\tau \to 0} \frac{o_k(\tau)}{\tau} = 0
\]

The third set of properties is really the crux of the process in the sense that they allow you to make some meaningful estimates. The "remainder" terms \( o(\tau) \) and \( o_k(\tau) \) are assumed to be negligible in comparison to \( \tau \) when \( \tau \) is very small. If one invests the process with sufficient differentiability, these terms can be thought of as \( O(\tau^2) \) terms in a Taylor series expansion of \( P(k, \tau) \). Thus for small \( \tau \) the probability of a single arrival is approximated by \( 1 - \lambda \tau \). The probability of two or more arrivals in a small interval is taken to be negligible in comparison to \( P(1, \tau) \) as \( \tau \) gets smaller.

Based on these assumptions we want to prove that \( P(n, t) = \frac{\lambda e^{-\lambda t}}{n!} \) for integral \( n \geq 1 \)

Consider \( P(n, t + h) \) which is the probability of \( n \) arrivals in the interval of length \( t + h \).

(i) Show that \( P(n, t + h) = P(n, t) (1 - \lambda h + o(h)) \)

(ii) Using (i) obtain a differential equation involving \( P(n, t) \). Does your equation hold for \( n = 0 \) and if not, why not?

(iii) Find the solution to the general differential equation \( \frac{dy}{dx} + p(x)y = q(x) \). Apply what you have derived to the Poisson problem for \( n = 1 \).

(iv) Now consider the differential equation \( \frac{dy}{dx} + p(x)y = q(x) \) and show that if \( q(x) \) is a solution of \( \frac{dy}{dx} + p(x)y = 0 \) then \( y(x) = x q(x) \) is a solution of \( \frac{dy}{dx} + p(x)y = q(x) \). Once again apply what you have derived to the Poisson problem for \( n = 1 \).

(v) After having done steps (i) -(iv) you should now know \( P(0, t) \) and \( P(1, t) \). Your next mission is to develop an inductive argument for \( P(n, t) \). Your goal is to show that \( P(n, t) = \frac{\lambda e^{-\lambda t}}{n!} \) for integral \( n \geq 1 \)

Solution

(i) There are three mutually exclusive ways the relevant event can occur and this is the "engine" for problems such as this. First, the process could have recorded \( n \) arrivals up to time \( t \) and then none in the interval \( (t, t+h] \). The probability of this is \( P(n, t)P(0, h) = P(n, t)[1 - \lambda h + o(h)] = P(n, t)[1 - \lambda h] + o(h) \)

There are two sub-steps here. Independence (property (2) above) has been used to assert that the required probability is \( P(n, t)P(0, h) \). The second step is to use property (3(a)) and note that \( P(n, t) o(h) \) is approximated by \( o(h) \). If the function \( o(h) \) were \( h^2 \) then as \( h \to 0, P(n, t) o(h) \) can be approximated by \( o(h) \)

The second possibility is that \( n \) arrivals have occurred up to time \( t \) and exactly one arrival occurs in \( (t, t+h] \). This probability is \( P(n-1, t)[\lambda h + o_1(\tau)] = P(n-1, t)\lambda h + o_1(\tau) \) using property (3(b)) and the same argument above in relation to \( P(n-1, t) o_1(\tau) \).

The third possibility is that there are \( n-2, n-3 \) etc arrivals up to time \( t \) and the corresponding arrivals in \( (t, t+h] \) are \( 2, 3 \) etc. The probability of such events is \( P(n-k, t)P(k, h) \) for \( k = 2, 3 \ldots \) But \( P(k, h) = o_k(\tau) \) from property (3(c)) and using the same reasoning above we see that the required probability for this event is \( o_k(\tau) \).

Putting all three components together we get:
\[
P(n, t + h) = P(n, t)(1 - \lambda h) + P(n - 1, t)\lambda h + o(h) \quad \text{(note here that the } o(h) \text{ terms have been aggregated but they are still } o(h) \}
\]

(ii) Note here that when \( n = 0 \) the second and third possibilities in the development do not apply and we would get:
\[
P(0, t + h) = P(0, t)(1 - \lambda h) + o(h) = P(0, t) - \lambda h P(0, t) + o(h)
\]

Therefore \( \frac{P(0, t + h) - P(0, t)}{h} = -\lambda P(0, t) + \frac{o(h)}{h} \) and as \( h \to 0 \) we have \( P'(0, t) = -\lambda P(0, t) \)

For \( n > 0 \) we proceed as follows:
\[
P(n, t + h) = P(n, t)(1 - \lambda h) + P(n - 1, t)\lambda h + o(h)
\]
\[
P(n, t + h) - P(n, t) = -\lambda h P(n, t) + P(n - 1, t)\lambda h + o(h)
\]

Hence \( \frac{P(n, t + h) - P(n, t)}{h} = -\lambda P(n, t) + P(n - 1, t)\lambda + \frac{o(h)}{h} \)

Now as \( h \to 0, \frac{o(h)}{h} \to 0 \) and the limit of the LHS exists and is \( P'(n, t) \). Although the development assumed \( h \) was positive that is not
necessary and o(h) does not depend on t and we could equally have used t - h in the development. This suggests that the last equation implies continuity. The other point is that the development might suggest a one-sided derivative (ie from the left) but one can prove, if pushed, that the derivative is a two sided one in the normal sense.

Squibbing the details we simply assert that
\[ P'(n, t) = -\lambda P(n, t) + \lambda P(n - 1, t) \]

(iii) The differential equation is one of the simplest to solve and if you already know how to do it you can skip over what follows. If not, here is the basic theory. To solve \( \frac{dy}{dx} + p(x) y = q(x) \) there are three basic steps.

(a) Find the general solution of the associated homogeneous equation which is \( \frac{dy}{dx} + p(x) y = 0 \)

(b) Find, either by inspired guessing or some systematic method, any solution of \( \frac{dy}{dx} + p(x) y = q(x) \).

(c) Add the two results.

Let's do the first step. Clearly \( y(x) = 0 \) is a solution to \( \frac{dy}{dx} + p(x) y = 0 \) so suppose \( y(x) \neq 0 \) so we can separate variables as follows:

\[ \frac{dy}{y} = -p(x) \, dx \]

Integrating we have
\[ \int \frac{dy}{y} = \int -p(x) \, dx + C \]
where \( C \) is a constant.

Therefore \( \ln |y| = -R(x) + C \) where \( R(x) = \int p(x) \, dx \)

So the solution is
\[ y = \begin{cases} K e^{-R(x)} & \text{if } y > 0 \\ -K e^{-R(x)} & \text{if } y < 0 \end{cases} \quad (K = e^C) \]

Our differential equation for \( n = 0 \) is solved as follows:

\[ P'(0, t) = -\lambda P(0, t) \]

Therefore
\[ \int P'(0, t) \, dt = \int -\lambda \, dt \]

ie \( \ln P(0, t) = -\lambda t + C \) (note that probabilities are always non-negative !)

\[ P(0, t) = e^{-\lambda t} e^{C} = e^{-\lambda t} \]

But \( P(0, 0) = 1 \) so \( 1 = K e^{0} = 1 \)

Thus our basic building block for what follows is \( P(0, t) = e^{-\lambda t} \)

(iv) If \( q(x) \) is a solution of \( \frac{dy}{dx} + p(x) y = 0 \) then \( y(x) = x q(x) \) is a solution of \( \frac{dy}{dx} + p(x) y = q(x) \).

The proof is nothing more than a calculation as follows:

\[ \frac{dy}{dx} + p(x) y = \frac{d}{dx} \left( x q(x) \right) + p(x) x q(x) \]

\[ = x \frac{dq}{dx} + q(x) + x p(x) q(x) \]

\[ = x \left[ \frac{dq}{dx} + p(x) q(x) \right] + q(x) \]

\[ = q(x) \quad \text{since} \quad \frac{dq}{dx} + p(x) q(x) = 0 \]

The form of our Poisson equation is : \( P'(n, t) = -\lambda P(n, t) + \lambda P(n - 1, t) \) and when \( n = 1 \) we have:

\[ \frac{dP(1, t)}{dt} + \lambda P(1, t) = \lambda e^{-\lambda t} \]

One might try something like \( P(1, t) = \lambda e^{-\lambda t} \) as a start so let's see where it gets us.
Thus demonstrates the relationship between the curve and the associated line segment for all \( x, y \). Not very far since we want the LHS to equal \( \lambda e^{-\lambda t} \). The most obvious next choice is \( P(1, t) = \lambda t e^{-\lambda t} \). Trying this we get:

\[
\frac{d}{dt}(\lambda t e^{-\lambda t}) + \lambda \lambda e^{-\lambda t} = -\lambda^2 e^{-\lambda t} + \lambda^2 e^{-\lambda t} = 0.
\]

Hence \( P(1, t) = \lambda t e^{-\lambda t} \) works as would \( P(1, t) = \lambda t e^{-\lambda t} + C \) where \( C \) is a constant. Note here that \( P(1, 0) = 0 \) (the probability of exactly one arrival at time zero is zero) so \( 0 = 0 + C \). Thus our solution is \( P(1, t) = \lambda t e^{-\lambda t} \).

(v) We know that \( P(0, t) = e^{-\lambda t} \) and \( P(1, t) = \lambda t e^{-\lambda t} \). The differential equation tells us that:

\[
P'(n, t) = -\lambda P(n, t) + \lambda P(n - 1, t)
\]

ie \( P'(n, t) + \lambda P(n, t) = \lambda P(n - 1, t) \)

Thus \( P(2, t) + \lambda P(2, t) = \lambda P(1, t) = \lambda^2 t e^{-\lambda t} \)

If we guessed \( (\lambda t)^2 e^{-\lambda t} \) as a solution we’d be nearly there since

\[
\frac{d}{dt}(\frac{1}{2} (\lambda t)^2 e^{-\lambda t}) + \lambda (\lambda t)^2 e^{-\lambda t} = -\lambda (\lambda t)^2 e^{-\lambda t} + e^{-\lambda t} \lambda^2 t + \lambda (\lambda t)^2 e^{-\lambda t} = e^{-\lambda t} \lambda^2 t
\]

This suggests that we should try \( \frac{(\lambda t)^2}{2!} e^{-\lambda t} \) and differentiation confirms the point:

\[
\frac{d}{dt}(\frac{1}{2!} (\lambda t)^2 e^{-\lambda t}) + \lambda (\lambda t)^2 e^{-\lambda t} = \lambda (\lambda t^2)^2 e^{-\lambda t} + e^{-\lambda t} \lambda^2 t + \lambda (\lambda t)^2 e^{-\lambda t} = e^{-\lambda t} \lambda^2 t
\]

Thus our inductive hypothesis is that \( P(n, t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \)

Now \( P(n + 1, t) + \lambda P(n + 1, t) = \lambda P(n, t) = \lambda \frac{(\lambda t)^n}{n!} e^{-\lambda t} \) using the induction hypothesis.

\[
P(n + 1, t) = \frac{(\lambda t)^{n+1}}{(n+1)!} e^{-\lambda t}
\]

is a solution of the differential equation since:

\[
\frac{d}{dt}(\frac{1}{(n+1)!} (\lambda t)^{n+1} e^{-\lambda t}) + \lambda \frac{(\lambda t)^{n+1}}{(n+1)!} e^{-\lambda t} = -\lambda \frac{(\lambda t)^{n+1}}{(n+1)!} e^{-\lambda t} + (n+1) \frac{\lambda (\lambda t)^n}{(n+1)!} e^{-\lambda t} + \lambda \frac{(\lambda t)^{n+1}}{(n+1)!} e^{-\lambda t}
\]

\[
= \frac{\lambda (\lambda t)^n}{n!} e^{-\lambda t}
\]

This establishes that \( P(n, t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \) is the solution for all integral \( n \geq 0 \).

**51. Proving the Cauchy-Schwarz inequality and much, much more using the properties of concave functions**

Recall that the Cauchy-Schwarz inequality is as follows:

Suppose \( a \) and \( b \) are positive real numbers for all \( i \) then \( \sum_{i=1}^{n} a_i b_i = \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2} \)

Now Cauchy proved this by writing \( \sum_{i=1}^{n} a_i b_i = \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2} \) in this equivalent form:

\[
0 = \sum_{s < j \leq n} (a_s b_j - a_j b_s)^2.
\]

There is, however, a much more powerful way of proving the Cauchy-Schwarz inequality and several other ones. The "engine" which enables this powerful generalisation relies upon the concept of strictly concave functions. A function \( g: \mathbb{R} \rightarrow \mathbb{R} \) is concave if it satisfies the following inequality:

\[
\lambda g(x) + (1 - \lambda) g(y) \leq g(\lambda x + (1-\lambda)y)
\]

for all \( x, y \in \mathbb{R} \) and for all real \( \lambda \) such that \( 0 < \lambda < 1 \). Geometrically this means that the line segment joining the points \( (x, g(x)) \) and \( (y, g(y)) \) lies below the graph of \( g \). Strict concavity occurs if equality in the above inequality is equivalent to \( x = y \). The following graph demonstrates the relationship between the curve and the associated line segment.
Problem

As a first step let's prove that \( g(x) = \sqrt{x} \) is strictly concave for \( x > 0 \). Thus we have to show that \( \lambda \sqrt{x} + (1 - \lambda) \sqrt{y} \leq \sqrt{\lambda x + (1 - \lambda) y} \) where \( 0 < \lambda < 1 \). After that is done we move onto a powerful generalisation which involves an inductive style of proof.

(i) First, show that for all \( x \) and \( y > 0 \), \( \sqrt{xy} \geq \frac{x + y}{2} \)

(ii) Using (i) show that \( \lambda \sqrt{x} + (1 - \lambda) \sqrt{y} \leq \sqrt{\lambda x + (1 - \lambda) y} \) where \( 0 < \lambda < 1 \)

(iii) Let \( g : \mathbb{R} \to \mathbb{R} \) be a strictly concave function and let \( f : \mathbb{R}^2 \to \mathbb{R} \) be the following function: \( f(x,y) = y \cdot g\left(\frac{x}{y}\right) \)

Then it follows that for all positive reals \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) satisfy this inequality:
\[
\sum_{i=1}^{n} f(x_i, y_i) \leq f\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\right)
\]
and equality holds if and only if the \( x_i \) and \( y_i \) are proportional.

Show this by an inductive proof on \( n \).

(iv) With all of this lead up, by selecting a suitable strictly concave function \( g(x) \) prove the Cauchy-Schwarz inequality
\[
\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}
\]
for positive \( a_i \) and \( b_i \)

Solution (i)

The solution is based on the observation that for any \( u, v: (u - v)^2 \geq 0 \)

Expanding this we have that \( u^2 + v^2 \geq 2uv \).

Now let \( u^2 = x \) and \( v^2 = y \) where \( x \) and \( y \) are positive (ignoring the trivial case where they are zero) and we then have:
\[
u = \sqrt{x} \quad \text{and} \quad v = \sqrt{y} \text{.}
\]

Hence \( \sqrt{xy} \geq \frac{x + y}{2} \) as claimed

If \( \sqrt{xy} = \frac{x + y}{2} \) we must have that \( u = v \) ie \( \sqrt{x} = \sqrt{y} \) so that \( x = y \), thereby establishing strict concavity.

Solution (ii)

Since we have established that \( \sqrt{xy} \geq \frac{x + y}{2} \) let's live dangerously and see if we can make something out of this:
\[
\lambda(1-\lambda) \cdot \sqrt{xy} \geq \lambda(1-\lambda) \cdot \frac{x + y}{2} \quad \text{where} \ 0 < \lambda < 1 . \quad \text{Note that} \ \lambda(1-\lambda) > 0 \ \text{so the direction of the inequality does not change.}
\]

Now make some inspired additions to both sides to get something that is of this form: \( (a + b)^2 \).
\[
\lambda^2 x + 2\lambda(1-\lambda) \sqrt{xy} + (1 - \lambda)^2 y \quad \lambda^2 x + \lambda(1-\lambda)(x + y) + (1 - \lambda)^2 y
\]
\[
\left( \lambda \sqrt{x} + (1 - \lambda) \sqrt{y} \right)^2 = \lambda^2 x + (1 - \lambda^2) \left[ \lambda x + \lambda y + (1 - \lambda)y \right] = \lambda^2 x + (1 - \lambda)(\lambda x + y) = \lambda x + (1 - \lambda)y
\]

Hence \( \lambda \sqrt{x} + (1 - \lambda) \sqrt{y} \) on taking square roots of both sides (both of which are positive)

This establishes the result, namely that, \( \sqrt{x} \) is concave for \( x > 0 \).

**Solution (iii)**

The case of \( n = 1 \) does not illuminate the way to proceed but the formula is clearly, indeed it is an equality and the two sequences of numbers are obviously proportional.

For \( n = 2 \) we need to use the concavity of \( g \) this way. Let \( \lambda = \frac{y_1}{y_1 + y_2} \) so that \( 1 - \lambda = \frac{y_2}{y_1 + y_2} \) then because \( g \) is concave we have:

\[
f(x_1, y_1) + f(x_2, y_2) = y_1 g \left( \frac{x_1}{y_1} \right) + y_2 g \left( \frac{x_2}{y_2} \right)
\]

\[
= \left( y_1 + y_2 \right) g \left( \frac{x_1 + x_2}{y_1 + y_2} \right)
\]

\[
= f(x_1 + x_2, y_1 + y_2) \text{ (using the definition of } f)\]

But \( f(x_1, y_1) + f(x_2, y_2) = y_1 g \left( \frac{x_1}{y_1} \right) + y_2 g \left( \frac{x_2}{y_2} \right) = y_1 g(t) + y_2 g(t) \) so we have equality in the relationship if the sequences are proportional.

Going in the other direction, if \( y_1 g \left( \frac{x_1}{y_1} \right) + y_2 g \left( \frac{x_2}{y_2} \right) = (y_1 + y_2) g \left( \frac{x_1 + x_2}{y_1 + y_2} \right) \) then using the strict concavity of \( g \) (which means that equality in \( \lambda g(x) + (1 - \lambda) g(y) \) \( g(\lambda x + (1-\lambda)y) \) is equivalent to \( x = y \)) we see that:

\[
\frac{y_1}{y_1 + y_2} g \left( \frac{x_1}{y_1} \right) + \frac{y_2}{y_1 + y_2} g \left( \frac{x_2}{y_2} \right) = g \left( \frac{x_1 + x_2}{y_1 + y_2} \right)
\]

and the LHS is in the form \( \lambda g(x) + (1 - \lambda) g(y) \) with \( \lambda = \frac{y_1}{y_1 + y_2} \). Strict concavity then implies that \( \frac{x_1}{y_1} = \frac{x_2}{y_2} = t \).

The induction hypothesis is that \( \sum_{i=1}^{n} f(x_i, y_i) \leq f \left( \sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i \right) \)

Now consider \( \sum_{i=1}^{n+1} f(x_i, y_i) = \sum_{i=1}^{n} f(x_i, y_i) + f(x_{n+1}, y_{n+1}) \)

\[
\leq f \left( \sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i \right) + f(x_{n+1}, y_{n+1}) \text{ using the induction hypothesis}
\]

\[
= \left( \sum_{i=1}^{n} y_i \right) \{ g \left( \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \right) + y_{n+1} g \left( \frac{x_{n+1}}{y_{n+1}} \right) \} \text{ using the definition of } f
\]

\[
\leq \left( \sum_{i=1}^{n+1} y_i \right) \{ g \left( \frac{\sum_{i=1}^{n+1} x_i}{\sum_{i=1}^{n+1} y_i} \right) + y_{n+1} g \left( \frac{x_{n+1}}{y_{n+1}} \right) \} \text{ using the concavity of } g \text{ and noting that } \lambda = \frac{y_{n+1}}{\sum_{i=1}^{n+1} y_i} \text{ and } 1 - \lambda = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n+1} y_i}
\]

\[
= f \left( \sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i \right) \text{ using the definition of } f \text{ again.}
\]

\[\text{ie } \sum_{i=1}^{n+1} f(x_i, y_i) \leq f \left( \sum_{i=1}^{n+1} x_i, \sum_{i=1}^{n+1} y_i \right)
\]

This establishes that the proposition is true for \( n + 1 \) and hence true for all \( n \).

Note that by the strict concavity of \( g \) we have that \( \sum_{i=1}^{n} g \left( \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \right) = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} = t \) (this comes from the assumed equality of

\[
\frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \text{ and } g \left( \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \right) + y_{n+1} g \left( \frac{x_{n+1}}{y_{n+1}} \right) + \frac{y_{n+1}}{\sum_{i=1}^{n+1} y_i} x_{n+1} \text{ and } g \left( \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} \right) + y_{n+1} g \left( \frac{x_{n+1}}{y_{n+1}} \right) + \frac{y_{n+1}}{\sum_{i=1}^{n+1} y_i} x_{n+1} \right)
\]

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But this will hold for any \( n \) ie we would also have that \( \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} = \frac{x_1}{y_1} \)

The reason for this is that we know this is that \( \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} = \frac{x_{n-1}}{y_{n-1}} = t \) holds quite generally for any choice of \( n \) of the sequence members forming the sums in the ratio. Just think of the sequence members being re-indexed.

If this does not convince you, then there is a more tedious way of demonstrating the point. Start with \( \sum_{i=1}^{n} x_i = t \sum_{i=1}^{n} y_i \) and note that \( \sum_{i=2}^{n} x_i = \sum_{i=1}^{n} x_i + x_{n-1} - x_1 \). We assume that \( \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} y_i} = \frac{x_1}{y_1} \) and ultimately want to show that this equals \( t \).

Now \( \sum_{i=2}^{n} x_i = \frac{x_1}{y_1} \sum_{i=2}^{n} y_i \) and so substituting for \( \sum_{i=2}^{n} x_i \) in the above we get:

\[
\frac{x_1}{y_1} \left( \sum_{i=2}^{n} y_i + y_{n-1} - y_1 \right) = t \sum_{i=2}^{n} y_i + x_{n-1} - x_1
\]

\[
\{ \frac{x_1}{y_1} - t \} \sum_{i=1}^{n} y_i + \frac{x_1}{y_1} y_{n-1} - x_1 = x_{n-1} - x_1
\]

\[
\{ \frac{x_1}{y_1} - t \} \sum_{i=1}^{n} y_i = x_{n-1} - \frac{x_1}{y_1} y_{n-1}
\]

\[
= t y_{n-1} - \frac{x_1}{y_1} y_{n-1}
\]

\[
= ( \frac{x_1}{y_1} - t ) y_{n-1}
\]

Therefore \( \{ \frac{x_1}{y_1} - t \} \sum_{i=1}^{n} y_i = 0 \) and since all the \( y \) are positive we must have that \( \frac{x_1}{y_1} = t \). This can be done for any other combination of the sequence members.

Hence the proposition is established for all \( n \) as is the claim of proportionality when there is equality of the functions.

Solution (iv)

We know that \( g(x) = \sqrt{x} \) is strictly concave so we see if we can use it in the form \( f(x, y) = y g \left( \frac{x}{y} \right) = y \sqrt{\frac{x}{y}} = \sqrt{xy} \)

The result proved in (iii) demonstrates that \( \sum_{i=1}^{n} x_i \sqrt{y_i} \sqrt{\sum_{i=1}^{n} x_i} \sqrt{\sum_{i=1}^{n} y_i} \)

By putting \( x_1 = a_1^2 \) and \( y_1 = b_1^2 \) the Cauchy inequality falls out and you get equality if the sequence members are proportional.

It can be shown that by suitable choice of concave function some very important inequalities can be established in a few lines. For more discussion see Gerhard J Woeginger, "When Cauchy and Hölder met Minkowski: A tour through well-known inequalities", Mathematics Magazine, Vol 82, No 3, June 2009 pages 202-207

52. Finding the pattern

The following table has a structure. Find it and prove your guess by induction:

<table>
<thead>
<tr>
<th>1</th>
<th>2 + 3 + 4</th>
<th>5 + 6 + 7 + 8 + 9</th>
<th>10 + 11 + 12 + 13 + 14 + 15 + 16</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>27</td>
<td>64</td>
</tr>
</tbody>
</table>

Solution

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The form of the RHS is easy enough to work out. If $S_n$ is the $n$th sum then $S_{n+1} = n^3 + (n + 1)^3$ for $n = 0, 1, 2, 3, \ldots$. The LHS is just sums of terms in arithmetic progression but you have to get the starting and end points right. The first term in each row is of the form: $n^2 + 1$ for $n = 0, 1, 2, 3, \ldots$. So we have our starting point. How many terms are there in each row? There is an odd number of terms and so we guess $2n+1$ terms which also looks right for $n = 0, 1, 2, 3, \ldots$.

Thus for the general $n+1$th row our guess would be:

\[
\frac{(n^2 + 1) + \sum_2 (n^2 + 2) + (n^2 + 3) + \ldots + (n + 1)^2}{n} = \frac{(n^2 + 1) + \sum_2 (n^2 + 2) + (n^2 + 3) + \ldots + (n + 1)^2}{n}.
\]

Note here that the final term simply reflects the arithmetical progression in the amounts added to $n^2$.

So what we have to establish by induction is whether:

\[
\sum_2 \frac{(n^2 + 1) + \sum_2 (n^2 + 2) + (n^2 + 3) + \ldots + (n + 1)^2}{n} = \frac{n^3 + (n + 1)^3}{n}.
\]

The base case is when $n = 0$ and we have equality of the LHS and RHS.

Assume that our formula is true for any integral $n \geq 0$; ie our induction hypothesis is that:

\[
\sum_2 \frac{(n^2 + 1) + \sum_2 (n^2 + 2) + (n^2 + 3) + \ldots + (n + 1)^2}{n} = \frac{n^3 + (n + 1)^3}{n}.
\]

Now consider the $n+1$ sum:\n
\[
P(n+1) = \sum_2 \frac{(n^2 + 1) + \sum_2 (n^2 + 2) + (n^2 + 3) + \ldots + (n + 1)^2}{n}.
\]

Now three are $2(n+1) + 1 = 2n + 3$ terms in $P(n+1)$. To use the inductive hypothesis one can write the terms as follows and simply count off the relevant components:

- \[n^2 + 2n + 1\]
- \[n^2 + 2n + 2\]
- \[\ldots\]
- \[n^2 + 2n + 2n + 1\]
- \[n^2 + 2n + 1 + 2n + 1\]

\[\text{the sum of the terms above this line is } n^3 + (n + 1)^3 + (2n + 1)^2\]

\[n^2 + 2n + 1 + 2n + 2\]

\[n^2 + 2n + 1 + 2n + 3\]

\[\text{the sum of the terms below the line is } 2n^2 + 8n + 7\]

Thus \[P(n+1) = n^3 + (n + 1)^3 + (2n + 1)^2 + 2n^2 + 8n + 7\]

\[= 2n^3 + 3n^2 + 3n + 1 + 4n^2 + 4n + 1 + 2n^2 + 8n + 7\]

\[= 2n^3 + 9n^2 + 15n + 9\]

Now \[(n + 1)^3 + (n + 2)^3 = n^3 + 3n^2 + 3n + 1 + n^3 + 6n^2 + 12n + 8\]

\[= 2n^3 + 9n^2 + 15n + 9\]

Hence \[P(n+1) = (n + 1)^3 + (n + 2)^3\] and thus our guess is established by induction.

51. The Tower of Hanoi

This is an ancient puzzle and consists of $n$ disks of decreasing diameters placed on a pole. There are two other poles. The problem is to move the entire pile to another pole by moving one disk at a time to any other pole, except that no disk may be placed on top of a smaller disk. Find a formula for the least number of moves needed to move $n$ disks from one pole to another and prove the formula by induction. This is a problem from Lindsay N Childs, "A Concrete Introduction to Higher Algebra", Second Edition, Springer, 2000 page 12.

The following diagram shows the set up.
Solution

An inductive way to proceed is to investigate some low order cases and see whether a suitable structure can be established.

When \( n = 1 \) there is one way of moving the one disk to another pole and clearly that is minimal.

When \( n = 2 \) there are three ways of performing the task in minimal terms as follows:

1. Step 1: \( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \end{array} \rightarrow \begin{array}{c} \mathbb{2} \\ \mathbb{1} \end{array} \left( \begin{array}{c} \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \right) \)
2. Step 2: \( \begin{array}{c} \mathbb{1} \\ \mathbb{1} \\ \mathbb{2} \end{array} \rightarrow \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \end{array} \left( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \right) \)
3. Step 2: \( \begin{array}{c} \mathbb{1} \\ \mathbb{1} \\ \mathbb{2} \end{array} \rightarrow \begin{array}{c} \mathbb{2} \\ \mathbb{1} \\ \mathbb{1} \end{array} \left( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \right) \)

When \( n = 3 \) there are seven ways of performing the tasks in minimal terms as follows:

1. Step 1: \( \begin{array}{c} \mathbb{2} \\ \mathbb{3} \\ \mathbb{1} \\ \mathbb{2} \end{array} \rightarrow \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{3} \\ \mathbb{1} \end{array} \left( \begin{array}{c} \mathbb{2} \\ \mathbb{3} \\ \mathbb{1} \\ \mathbb{2} \end{array} \right) \)
2. Step 2: \( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{3} \\ \mathbb{1} \end{array} \rightarrow \begin{array}{c} \mathbb{1} \\ \mathbb{3} \\ \mathbb{2} \\ \mathbb{1} \end{array} \left( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{3} \\ \mathbb{1} \end{array} \right) \)
3. Step 3: \( \begin{array}{c} \mathbb{1} \\ \mathbb{3} \\ \mathbb{1} \\ \mathbb{2} \end{array} \rightarrow \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \left( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \right) \)
4. Step 4: \( \begin{array}{c} \mathbb{1} \\ \mathbb{3} \\ \mathbb{1} \\ \mathbb{2} \end{array} \rightarrow \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \left( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \right) \)
5. Step 5: \( \begin{array}{c} \mathbb{1} \\ \mathbb{3} \\ \mathbb{1} \\ \mathbb{2} \end{array} \rightarrow \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \left( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \right) \)
6. Step 6: \( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \rightarrow \begin{array}{c} \mathbb{1} \\ \mathbb{3} \\ \mathbb{1} \\ \mathbb{2} \end{array} \left( \begin{array}{c} \mathbb{1} \\ \mathbb{3} \\ \mathbb{1} \\ \mathbb{2} \end{array} \right) \)
7. Step 7: \( \begin{array}{c} \mathbb{1} \\ \mathbb{2} \\ \mathbb{1} \\ \mathbb{2} \end{array} \rightarrow \begin{array}{c} \mathbb{1} \\ \mathbb{3} \\ \mathbb{1} \\ \mathbb{2} \end{array} \left( \begin{array}{c} \mathbb{1} \\ \mathbb{3} \\ \mathbb{1} \\ \mathbb{2} \end{array} \right) \)

It is beginning to look like the required formula is \( 2^n - 1 \) but let's do \( n = 4 \) just to see whether our guess still holds good.

We will come back to some structural observations later.
Step 1: \[
\begin{pmatrix}
2 \\
3 \\
4
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 2: \[
\begin{pmatrix}
3 \\
4
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 3: \[
\begin{pmatrix}
3 \\
4
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 4: \[
\begin{pmatrix}
4
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 5: \[
\begin{pmatrix}
1 \\
4
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 6: \[
\begin{pmatrix}
1 \\
4
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 7: \[
\begin{pmatrix}
4
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 8: \[
\begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \rightarrow \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 9: \[
\begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \rightarrow \begin{pmatrix}
2 \\
3 \\
1
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 10: \[
\begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 11: \[
\begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ \\
\circ
\end{pmatrix}
\]

Step 12: \[
\begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix} \rightarrow \begin{pmatrix}
\circ \\
\circ \\
\circ \\
\circ
\end{pmatrix} \begin{pmatrix}
\circ \\
\circ \\
\circ \\
\circ
\end{pmatrix}
\]
Step 13: \[
\begin{pmatrix}
\square \\
\square \\
2
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
3
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
4
\end{pmatrix}
\]

Step 14: \[
\begin{pmatrix}
\square \\
\square \\
\square \\
\square
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
2
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
3
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
4
\end{pmatrix}
\]

Step 15: \[
\begin{pmatrix}
\square \\
\square \\
\square \\
\square
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
1
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
2
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
3
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
4
\end{pmatrix}
\]

So it seems that the formula \(2^n - 1\) holds for \(n = 4\) too since \(2^4 - 1 = 15\).

In order to develop an inductive proof of our guess we need to consider a recursive expression for \(S(n)\) as follows. Consider a tower of \(n+1\) disks then \(S(n+1)\) can be determined by removing the top disk leaving a tower of \(n\) disks thus:

\[
\begin{pmatrix}
2 \\
\cdot \\
\cdot \\
n + 1
\end{pmatrix} \rightarrow \begin{pmatrix}
\square \\
\square \\
\square \\
\square
\end{pmatrix} + \text{ghost column}
\]

Assume there is a "ghost" fourth column which is invented to take account of the fact that one disk has already been put in one of the right hand columns. There are \(S(n)\) ways of getting the far left hand column of \(n\) disks into the far right hand and ghost columns if we ignore the middle column (in effect this is our induction hypothesis with \(n\) disks). This gives \(1 + S(n)\) moves. But each of the \(S(n)\) moves into the ghost column have to be transacted into one or other of the three real columns, giving rise to another \(S(n)\) moves. In effect this reverses the use of the ghost column. Thus \(S(n+1) = 1 + S(n) + S(n) = 1 + 2S(n)\).

As a check, \(S(1) = 1 + 2S(0) = 1\) (there is only one way of moving one disk).
\[
S(2) = 1 + 2S(1) = 1 + 2.1 = 3
\]
\[
S(3) = 1 + 2S(2) = 1 + 2.3 = 7
\]
\[
S(4) = 1 + 2S(3) = 1 + 2.7 = 15\quad\text{as before.}
\]

Following the chain of relationships through we get:

\[
S(n+1) = 1 + 2S(n)
\]
\[
= 1 + 2 \times (1 + 2S(n-1))
\]
\[
= 1 + 2 + 2^2 \times S(n-1)
\]
\[
= 1 + 2 + 2^2 \times (1 + 2S(n-2))
\]
\[
= 1 + 2 + 2^2 + 2^3 \times S(n-2)
\]
\[
\ldots
\]
\[
= 1 + 2 + 2^2 + 2^3 + \ldots + 2^nS(1)
\]
\[
= 1 + 2 + 2^2 + 2^3 + \ldots + 2^n \times 1\quad\text{(since we know that } S(1) = 1)\]
\[
= 2^{n+1} - 1\quad\text{which has the correct form.}
\]

Our guess was \(S(n) = 2^n - 1\) so that \(S(n+1) = 1 + 2S(n) = 1 + 2 \times (2^n - 1) = 2^{n+1} - 1\) as before. Thus our guess is established by inductive principles. \(S(n) = 2^n - 1\) must be the minimal number because from the design of the problem there are only two ways at any point of making a choice.

There are other ways of looking at this problem. One it to note that at each step there are two ways to place a disk so that for \(n\) disks the number of ways in which a selection can be made is simply \(2^n\) but this includes the case of never doing anything at any stage, hence the required number of ways is \(2^n - 1\).

**APPENDIX**

**Sup and inf**

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If $S$ is a non-empty subset of $\mathbb{R}$ and if every element of $S$ is less than or equal to $b$, then $b$ is called an upper bound of $S$. If $S$ has an upper bound then we say it is bounded above.

Similarly, if $S$ is a non-empty subset of $\mathbb{R}$ and if every element of $S$ is greater than or equal to $c$, then $c$ is called a lower bound of $S$. If $S$ has a lower bound then we say it is bounded below.

If $S$ is bounded above and below it is called bounded.

The real number $b$ is a supremum or least upper bound of $S$ if:

(a) $b$ is an upper bound of $S$; and
(b) there is no upper bound of $S$ less than $b$.

We write $b = \sup S$ (or lub $S$)

Symmetrically, the real number $c$ is an infimum or greatest lower bound of $S$ if:

(a) $c$ is a lower bound of $S$; and
(b) there is no lower bound of $S$ greater than $c$

We write $c = \inf S$ (or glb $S$)

Basics of partial fractions

This is a quick introduction to partial fractions if you have forgotten the theory. Let $a(x)$ be a polynomial of degree $1$ whose coefficients belong to some field $F$, say $\mathbb{R}$ or $\mathbb{C}$. If $a(x) = b(x)c(x)$ where $b(x)$ and $c(x)$ are polynomials also with coefficients in $F$ and both have degree $1$ we say that $a(x)$ is reducible over $F$ and that $b(x)$ and $c(x)$ are factors or divisors of $a(x)$. However, if the representation $a(x) = b(x)c(x)$ with coefficients in $F$ is only possible by taking $b(x)$ or $c(x)$ to be a constant (ie a polynomial of degree 0 over $F$), $a(x)$ is said to be prime or irreducible in $F[x]$. For instance, $x^2 + 1$ is irreducible over $\mathbb{R}$ but not over $\mathbb{C}$ since $(x + i)(x - i)$.

The basic theory (which I will not prove) is that if $a(x)$ and $b(x)$ are relatively prime polynomials over $F$ and have degrees of $m$ and $n$ respectively, then every polynomial $f(x)$ over $F$ of degree less than $m + n$ has a unique representation:

$$f(x) = B(x) a(x) + A(x) b(x)$$

where $A(x)$ and $B(x)$ are polynomials over $F$ of degree less than $m$ and $n$ respectively.

This means that we can write

$$\frac{f(x)}{a(x) b(x)} = \frac{A(x)}{a(x)} + \frac{B(x)}{b(x)}$$

Example: Find real numbers $A$, $B$, $C$ such that

$$\frac{x + 9}{x(x^2 - 9)} = \frac{A}{x} + \frac{Bx + C}{x^2 - 9}$$

Multiplying through by $x(x^2 - 9)$ we have that $x + 9 = A(x^2 - 9) + x(Bx + C) = (A + B)x^2 + Cx - 9A$

Equating coefficients we have that $C = 1$, $A = -1$ and $B = 1$ so that

$$\frac{x + 9}{x(x^2 - 9)} = \frac{-1}{x} + \frac{x + 1}{x^2 - 9}$$