Lagrange's vector cross product identity

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1 Introduction

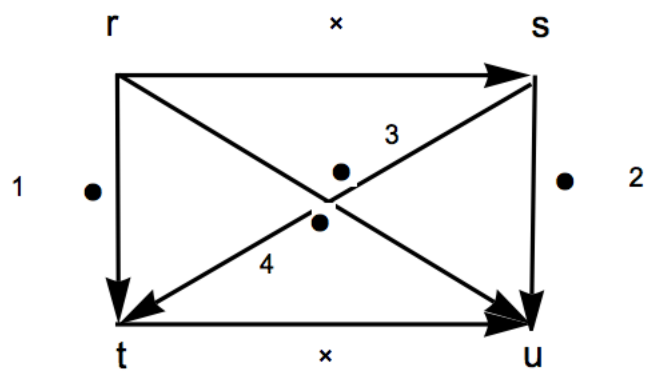
Suppose you are doing vector calculus and you have forgotten either one or both of the following two identities:

$$(\mathbf{r}\times\mathbf{s})\,.\,(\mathbf{t}\times\mathbf{u}) = (\mathbf{r}\,.\,\mathbf{t})\,(\mathbf{s}\,.\,\mathbf{u}) - (\mathbf{r}\,.\,\mathbf{u})\,(\mathbf{s}\,.\,\mathbf{t}) \tag{1}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\,.\,\mathbf{c}) - \mathbf{c}(\mathbf{a}\,.\,\mathbf{b}) \tag{2}$$

(1) is often called Lagrange's identity for 4 vectors or some variation of that. (2) is often referred to as the vector triple product rule. How do you remember (1) and (2)? Of the two, (2) is easier because of the acronym "BAC - CAB" but you still have to remember that the "AC" and "AB" parts reflect dot products.

Remembering (1) is harder. especially if it is not used frequently. The diagram below may assist:



2 Proofs

To prove (1) all you need to do is remember the definition of the cross product and be prepared to perform some straightforward if somewhat tedious algebra. You only need to do this once in your life.

The cross product can be presented as follows:

$$\mathbf{r}\times\mathbf{s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = \begin{vmatrix} r_2 & r_3 \\ s_2 & s_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} r_3 & r_1 \\ s_3 & s_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix}\mathbf{k} \tag{3}$$

Note here that the minus sign for the expansion in column 2 is incorporated in the determinant. You may be used to writing the expansion as follows to explicitly recognise the sign:

$$\mathbf{r}\times\mathbf{s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = \begin{vmatrix} r_2 & r_3 \\ s_2 & s_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} r_1 & r_3 \\ s_1 & s_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix}\mathbf{k} \tag{4}$$

Similarly, we have:

$$\mathbf{t}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t_1 & t_2 & t_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \begin{vmatrix} t_2 & t_3 \\ u_2 & u_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} t_3 & t_1 \\ u_3 & u_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} t_1 & t_2 \\ u_1 & u_2 \end{vmatrix}\mathbf{k} \tag{5}$$

The dot product is:

$$
(\mathbf{r}\times\mathbf{s})\cdot(\mathbf{t}\times\mathbf{u}) = \begin{pmatrix} \begin{vmatrix} r_2 & r_3 \\ s_2 & s_3 \end{vmatrix} \\ \\ \begin{vmatrix} r_3 & r_1 \\ s_3 & s_1 \end{vmatrix} \\ \\ \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix} \end{pmatrix} \cdot \begin{pmatrix} \begin{vmatrix} t_2 & t_3 \\ u_2 & u_3 \end{vmatrix} \\ \\ \begin{vmatrix} t_3 & t_1 \\ u_3 & u_1 \end{vmatrix} \\ \\ \begin{vmatrix} t_1 & t_2 \\ u_1 & u_2 \end{vmatrix} \end{pmatrix}
$$

$$
\begin{aligned}
&= \begin{vmatrix} r_2 & r_3 \\ s_2 & s_3 \end{vmatrix} \begin{vmatrix} t_2 & t_3 \\ u_2 & u_3 \end{vmatrix} + \begin{vmatrix} r_3 & r_1 \\ s_3 & s_1 \end{vmatrix} \begin{vmatrix} t_3 & t_1 \\ u_3 & u_1 \end{vmatrix} + \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix} \begin{vmatrix} t_1 & t_2 \\ u_1 & u_2 \end{vmatrix} \\
&= (r_2 s_3 - s_2 r_3)(t_2 u_3 - u_2 t_3) + (r_3 s_1 - s_3 r_1)(t_3 u_1 - u_3 t_1) + (r_1 s_2 - s_1 r_2)(t_1 u_2 - u_1 t_2) \\
&= \quad r_2 s_3 t_2 u_3 - r_2 s_3 t_3 u_2 - r_3 s_2 t_2 u_3 + r_3 s_2 t_3 u_2 \\
&\quad + r_3 s_1 t_3 u_1 - r_3 s_1 t_1 u_3 - r_1 s_3 t_3 u_1 + r_1 s_3 t_1 u_3 \\
&\quad + \underbrace{r_1 s_2 t_1 u_2}_{1} - \underbrace{r_1 s_2 t_2 u_1}_{3} - \underbrace{r_2 s_1 t_1 u_2}_{4} + \underbrace{r_2 s_1 t_2 u_1}_{2}
\end{aligned} \tag{6}
$$

We now add the terms in the columns labelled 1 and 2:

$$
\begin{aligned}
\text{sum of columns 1 and 2} &= r_1 t_1 (s_3 u_3 + s_2 u_2) + r_2 t_2 (s_3 u_3 + s_1 u_1) + r_3 t_3 (s_2 u_2 + s_1 u_1) \\
&= r_1 t_1 (\mathbf{s}\cdot\mathbf{u}) - r_1 s_1 t_1 u_1 + r_2 t_2 (\mathbf{s}\cdot\mathbf{u})) - r_2 s_2 t_2 u_2 + r_3 t_3 (\mathbf{s}\cdot\mathbf{u})) - r_3 s_3 t_3 u_3 \\
&= (\mathbf{r}\cdot\mathbf{t})(\mathbf{s}\cdot\mathbf{u}) - \sum_{i=1}^{3} r_i s_i t_i u_i
\end{aligned} \tag{7}
$$

Similarly we add the terms in columns labelled 3 and 4:

$$
\begin{aligned}
\text{sum of columns 3 and 4} &= -\Big[s_1 t_1 (r_3 u_3 + r_2 u_2) + s_2 t_2 (r_3 u_3 + r_1 u_1) + s_3 t_3 (r_2 u_2 + r_1 u_1) \Big] \\
&= -\Big[s_1 t_1 (\mathbf{r}\cdot\mathbf{u})) - r_1 s_1 t_1 u_1 + s_2 t_2 (\mathbf{r}\cdot\mathbf{u})) - r_2 s_2 t_2 u_2 + s_3 t_3 (\mathbf{u}\cdot\mathbf{u})) - r_3 s_3 t_3 u_3 \Big] \\
&= -(\mathbf{s}\cdot\mathbf{t})(\mathbf{r}\cdot\mathbf{u}) + \sum_{i=1}^{3} r_i s_i t_i u_i
\end{aligned} \tag{8}
$$

By adding equations (7) and (8) we get equation (1).

We can prove (1) a slightly different way by noting that:

$$
(\mathbf{r}\times\mathbf{s})\cdot(\mathbf{t}\times\mathbf{u}) = \mathbf{a}\cdot(\mathbf{t}\times\mathbf{u}) \tag{9}
$$

where $\mathbf{a} = \mathbf{r}\times\mathbf{s}$.

The righthand side of (9) is the volume of a parallelepiped with sides $\mathbf{a}, \mathbf{r}, \mathbf{s}$. That volume is reprsented by this determinant:

$$\begin{vmatrix} a_1 & t_1 & u_1 \\ a_2 & t_2 & u_2 \\ a_3 & t_3 & u_3 \end{vmatrix} \tag{10}$$

where:

$$a_1 = \begin{vmatrix} r_2 & r_3 \\ s_2 & s_3 \end{vmatrix} = r_2 s_3 - s_2 r_3 \tag{11}$$

$$a_2 = \begin{vmatrix} r_3 & r_1 \\ s_3 & s_1 \end{vmatrix} = r_3 s_1 - s_3 r_1 \tag{12}$$

$$a_3 = \begin{vmatrix} r_1 & r_2 \\ s_1 & s_2 \end{vmatrix} = r_1 s_2 - s_1 r_2 \tag{13}$$

Therefore:

$$\begin{aligned} \begin{vmatrix} a_1 & t_1 & u_1 \\ a_2 & t_2 & u_2 \\ a_3 & t_3 & u_3 \end{vmatrix} = &\begin{vmatrix} r_2 s_3 - s_2 r_3 & t_1 & u_1 \\ r_3 s_1 - s_3 r_1 & t_2 & u_2 \\ r_1 s_2 - s_1 r_2 & t_3 & u_3 \end{vmatrix} \\ =&(r_2 s_3 - s_2 r_3)(t_2 u_3 - t_3 u_2) - (r_3 s_1 - s_3 r_1)(t_1 u_3 - t_3 u_1) + (r_1 s_2 - s_1 r_2)(t_1 u_2 - t_2 u_1) \\ =&r_2 s_3 t_2 u_3 - r_2 s_3 t_3 u_2 - r_3 s_2 t_2 u_3 + r_3 s_2 t_3 u_2 - r_3 s_1 t_1 u_3 + r_3 s_1 t_3 u_1 + r_1 s_3 t_1 u_3 - r_1 s_3 t_3 u_1 \\ &+ r_1 s_2 t_1 u_2 - r_1 s_2 t_2 u_1 - r_2 s_1 t_1 u_2 + r_2 s_1 t_2 u_1 \end{aligned} \tag{14}$$

By inspection (14) can be seen to equal (6).

To prove (2) we let $\mathbf{d} = \mathbf{b} \times \mathbf{c}$ hence we need to expand:

$$\mathbf{a} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \tag{15}$$

Now:

$$\begin{aligned} \mathbf{d} =& \mathbf{b} \times \mathbf{c} \\ =& \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ =& \underbrace{(b_2 c_3 - c_2 b_3)}_{d_1} \mathbf{i} + \underbrace{(b_3 c_1 - c_3 b_1)}_{d_2} \mathbf{j} + \underbrace{(b_1 c_2 - c_1 b_2)}_{d_3} \mathbf{k} \end{aligned} \tag{16}$$

Therefore:

$$
\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) =& \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - c_2b_3 & b_3c_1 - c_3b_1 & b_1c_2 - c_1b_2 \end{vmatrix} \\
=&(a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3)]\mathbf{i} - (a_1b_1c_2 - a_1b_2c_1 - a_3b_2c_3 + a_3b_3c_2)\mathbf{j}+ \\
&(a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2)\mathbf{k}
\end{aligned} \tag{17}
$$

Now we look at the relevant components of:

$$\mathbf{b}(\mathbf{a}\,.\,\mathbf{c}) - \mathbf{c}(\mathbf{a}\,.\,\mathbf{b}) \tag{18}$$

The **i** component of (18) is:

$$
\begin{aligned}
b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3) =& a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1 \\
=& a_2b_1c_2 + a_3b_1c_3 - a_2b_2c_1 - a_3b_3c_1
\end{aligned} \tag{19}
$$

Comparison of (19) with the **i** component of (17) demonstrates they are equal. The process is repeated for the remaining components.

The **j** component of (18) is:

$$
\begin{aligned}
b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3) =& a_1b_2c_1 + a_2b_2c_2 + a_3b_2c_3 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_2 \\
=& a_1b_2c_1 + a_3b_2c_3 - a_1b_1c_2 - a_3b_3c_2
\end{aligned} \tag{20}
$$

Again (20) equals the **j** component of (17).

Finally, the The **k** component of (18) is:

$$
\begin{aligned}
b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3) =& a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 - a_1b_1c3 - a_2b_2c_3 - a_3b_3c_3 \\
=& a_1b_3c_1 + a_2b_3c_2 - a_1b_1c_3 - a_2b_2c_3
\end{aligned} \tag{21}
$$

This component also equals the **k** component of (17) and (2) is proved.

There are "shorter" proofs but when you analyse them they are built upon several preliminary results that are needed to shorten the final step. For instance in [1, Problem 48, page 29] equation (2) and some other preliminary results are used to prove (1) in a few lines.

3 References

[1] Murray R Spiegel, *"Theory and Problems of Vector Analysis and an introduction to Tensor Analysis"*,Schaum's Outline Series, McGraw Hill, 1974.

4 History

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