Multiple applications of L'Hôpital's rule

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1 Introduction

L'Hôpital's rule is one of mathematics' most famous bought titles. It originally appeared in L'Hôpital's 1696 text titled "Analyse des infiniment petits". However, Johann Bernoulli discovbered the rule circa 1694 when he weas employed by the Marquis to provide lectures on calculus. In a letter to Bernoulli L'Hôpital made him an offer as follows:

"I shall give you with pleasure a pension of three hundred livres...I ask you to give me occasionally some hours of your time to work on what I shall ask you - and also to communicate to me your discoveries with the request not to mention them to others." [1]

So he bought the rights to pass off Bernoulli's invention as his own and saddles him with a non-disclosure clause! L'Hôpital did later refer to his liberal use of others' intellectual property in the introduction to his textbook. In what follows I draw upon William Dunham's article [1] which shows how Euler used L'Hôpital's rule in various situations.

2 Problems requiring multiple uses of the rule

Euler used L'Hôpital's rule to solve this problem: "Find the value of the expression $\frac{x^x - x}{1 - x + \ln x}$ when we place $x = 1$ ".

Recall that L'Hôpital's rule is this (noting here that when $x = 1$ we will have the form $\frac{0}{0}$) :

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{'}(x)}{g^{'}(x)} \tag{1}$$

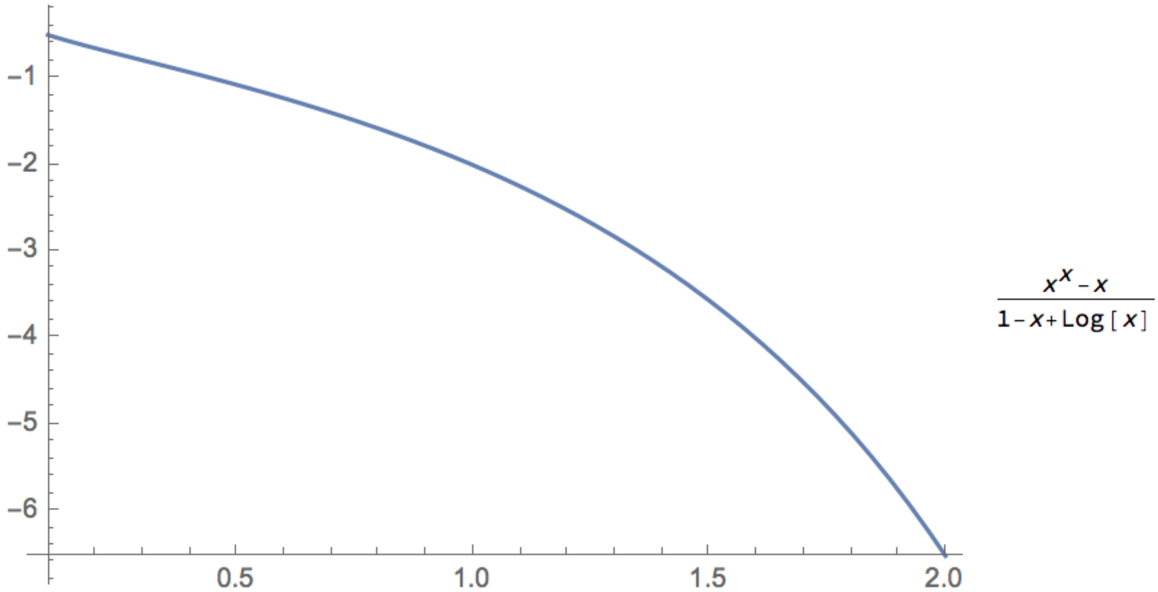
With $f(x) = x^x - x = e^{x\ \ln x} - x$ and $g(x) = 1 - x + \ln x$ we have that:

$$\begin{aligned} \lim_{x \to 1} \frac{f^{'}(x)}{g^{'}(x)} &= \lim_{x \to 1} \frac{x^x(1 + \ln x) - 1}{-1 + \frac{1}{x}} \\ &= \frac{0}{0} \end{aligned} \tag{2}$$

Faced with this Euler applies the rule a second time:

$$\begin{aligned}\lim_{x\to} \frac{x^x(1+\ln x)-1}{-1+\frac{1}{x}} =& \lim_{x\to 1} \frac{x^x(1+\ln x)^2 + x^{x-1}}{-\frac{1}{x^2}} \\ =& -2\end{aligned} \tag{3}$$

We can use Mathematica or Matlab to plot the relevant ratio and it shows us that it does indeed approach -2 as $x \to 1$ (from both directions).



3 Can we get the limit without 'Hôpital's rule ?

For $\epsilon > 0$ let us consider the left hand limit ie $x = 1-\epsilon \to 1$ as $\epsilon \to 0$. Then:

$$\begin{aligned}x^x - x =& e^{(1-\epsilon)\ln(1-\epsilon)} - (1-\epsilon) \\ \approx& 1 + (1-\epsilon)\ln(1-\epsilon) + \frac{1}{2!}[(1-\epsilon)\ln(1-\epsilon)]^2 - 1 + \epsilon \\ =& \epsilon + (1-\epsilon)\ln(1-\epsilon) + \frac{1}{2}[(1-\epsilon)\ln(1-\epsilon)]^2\end{aligned} \tag{4}$$

where Taylor's theorem has been used to approximate the function.

We need to use Taylor's theorem to get a second order approximation of $\ln(1\pm\epsilon)$ (to cover limits from both sides).

If $f(x) = \ln x$ then Taylors's theorem allows us to write for $0 \le x < 1$ to second order:

$$f(1-x) \approx f(1) + (-x)f^{'}(1) + (-x)^2 \frac{f^{''}(1)}{2!} \tag{5}$$

But $f^{'}(x) = \frac{1}{x}$ and $f^{''}(x) = -\frac{1}{x^2}$ so (6) becomes:

$$f(1-x) \approx -x - \frac{x^2}{2} \tag{6}$$

Using (6) we have the following:

$$\begin{aligned} 1 - x + \ln x =& 1 - (1-\epsilon) + \ln(1-\epsilon) \\ \approx& \epsilon - \epsilon - \frac{\epsilon^2}{2} \\ =& -\frac{\epsilon^2}{2} \end{aligned} \tag{7}$$

We now use (7) to expand (4) as follows:

$$\begin{aligned} \epsilon + (1-\epsilon)\ln(1-\epsilon) + \frac{1}{2}[(1-\epsilon)\ln(1-\epsilon)]^2 \approx& \epsilon + (1-\epsilon)(-\epsilon - \frac{\epsilon^2}{2}) + \frac{\epsilon^2}{2}(1-\epsilon)^2(1+\epsilon+\frac{\epsilon^2}{4}) \\ =& \epsilon - \epsilon - \frac{\epsilon^2}{2} + \epsilon^2 + \frac{\epsilon^3}{2} + \frac{\epsilon^2}{2}(1 - 2\epsilon + \epsilon^2)(1 + \epsilon + \frac{\epsilon^2}{4}) \\ =& \frac{\epsilon^2}{2} + \frac{\epsilon^3}{2} + \frac{\epsilon^2}{2}\left(1 + \epsilon + \frac{\epsilon^2}{4} - 2\epsilon - 2\epsilon^2 - \frac{\epsilon^3}{2} + \epsilon^2 + \epsilon^3 + \frac{\epsilon^4}{4}\right) \\ =& \epsilon^2 - \frac{3\epsilon^4}{8} + \frac{\epsilon^5}{4} + \frac{\epsilon^6}{8} \end{aligned} \tag{8}$$

Hence the limit becomes:

$$\begin{aligned} \lim_{x\to 1} \frac{x^x - x}{1 - x + \ln x} =& \lim_{\epsilon \to 0} \frac{\epsilon^2 - \frac{3\epsilon^4}{8} + \frac{\epsilon^5}{4} + \frac{\epsilon^6}{8}}{-\frac{\epsilon^2}{2}} \\ =& -2 \end{aligned} \tag{9}$$

The right hand limit taken with $x = 1+\epsilon$ as $\epsilon \to 0$ is done similarly and the limit is also -2.

William Dunham explains how Euler used L'Hôpital's rule not once but three times to prove his famous formula:

$$\sum_{n=1}^{\infty} = \frac{\pi^2}{6} \tag{10}$$

4 References

[1] William Dunham, "When Euler met L'Hôpital", Mathematics Magazine, Mathematical Association of America, Vol 82, No 1, February 2009, pages 16-25

5 History

Created

10/08/2018