Riemann integration: some practical examples using your bare hands

Peter Haggstrom
www.gotohaggstrom.com
mathsatbondibeach@gmail.com

October 22, 2012

1 Introduction

In calculus and analysis courses there are the inevitable examples of Riemann integration to convince you that the abstract definition actually gives the results you get when you apply the well known rules of integral calculus. It is like using Maxwell's equations to prove Kirchoff's Laws for a basic circuit - you do it once in your life to prove the concept and never repeat the experience! In this article I go through one simple and two more difficult examples to demonstrate that you do indeed get the right results by applying the basic Riemannian definition. The examples emphasise algebraic manipulations and basic limit theory. They are exercises in G.H Hardy, "A Course of Pure Mathematics", Cambridge University Press, Tenth Edition, 2006 pages 319-320.

Recall from the theory that you make a partition of the domain (x-axis) which does not necessarily have to be an equal subdivision of the relevant interval and you refine that partition by a limiting process (ie making the partitions ever smaller). If the theory works (and it does for garden variety functions) you should get the same answer for the limit no matter which type of partition you take. Indeed, this is what is proved at a general level in the theoretical treatment. In what follows we put theory into practice.

2 Simple example

The idea is to integrate $I = \int_a^b x dx$ by dividing [a,b] where into n equal sub-intervals and forming rectangles, summing them and then letting $n \to \infty$. The height of the rectange is $f(x_i) = x_i$ while the base has width $x_{i+1} - x_i$. Hence $x_i = a + i\frac{(b-a)}{n}$ so that $x_o = a$ and $x_n = b$. We know the answer is $\frac{b^2-a^2}{2}$.

Our approximation to I is:

$$I \approx \sum_{i=0}^{n-1}(x_{i+1}-x_i)f(x_i) = \sum_{i=0}^{n-1}\frac{b-a}{n}\left(a+i\frac{(b-a)}{n}\right) = \frac{b-a}{n}\sum_{i=0}^{n-1}\left(a+i\frac{(b-a)}{n}\right)$$
$$= \frac{b-a}{n}\left(na+\frac{(b-a)}{n}\frac{n(n-1)}{2}\right) = (b-a)a+(b-a)^2\left(\frac{1}{2}-\frac{1}{2n}\right) \quad (1)$$

Now as $n \to \infty$, $(b-a)a+(b-a)^2\left(\frac{1}{2}-\frac{1}{2n}\right) \to (b-a)a+\frac{(b-a)^2}{2} = \frac{(b-a)(b-a+2a)}{2} = \frac{b^2-a^2}{2}$. So we get the result we are after.

If we approximate the integral by $\sum_{i=0}^{n-1}(x_{i+1}-x_i)f(x_{i+1})$ the final result is $\frac{b^2-a^2}{2}(1-\frac{1}{n})+\frac{b(b-a)}{n}$ which also converges to $\frac{b^2-a^2}{2}$ as $n \to \infty$.

3 Example with geometric sub-intervals

In this example instead of the base of the rectangles being equal (ie arising from a linear decompositon of the interval) we split the interval [a,b], where $0 < a < b$, into geometrically related portions as follows:

$x_0 = a$, $x_1 = ar$, $x_2 = ar^2$,...,$x_i = ar^i \ldots, x_n = ar^n = b$ so that $r^n = \frac{b}{a}$.

Proceeding as before we get:

$$\int_a^b xdx \approx \sum_{i=0}^{n-1}(x_{i+1}-x_i)f(x_i) = \sum_{i=0}^{n-1}(ar^{i+1}-ar^i)ar^i = a^2(r-1)\sum_{i=0}^{n-1}r^{2i} = a^2(r-1)\frac{1-r^{2n}}{1-r^2}$$
$$= -a^2\frac{1-r^{2n}}{1+r} = \frac{-a^2(1-(\frac{b}{a})^2)}{1+(\frac{b}{a})^{\frac{1}{n}}} = \frac{b^2-a^2}{1+(\frac{b}{a})^{\frac{1}{n}}} \quad (2)$$

But since $\frac{b}{a} > 1$, $\sqrt[n]{\frac{b}{a}} \to 1$ as $n \to \infty$. Hence the sum approaches $\frac{b^2-a^2}{2}$ as before. If you cannot remember how to prove that $\sqrt[n]{\frac{b}{a}} \to 1$, consider the following:

Let $y = 1 + x_n$ where $x_n > 0$ for all n. Then using the binomial theorem $(1+x_n)^{\frac{1}{n}} > 1 + \frac{1}{n}x_n$. Hence as $n \to \infty$, $\sqrt[n]{y} \to 1$ since $\frac{x_n}{n} \to 0$.

Now let's go through the same process for $\int_a^b x^m dx = \frac{b^{m+1}-a^{m+1}}{m+1}$.

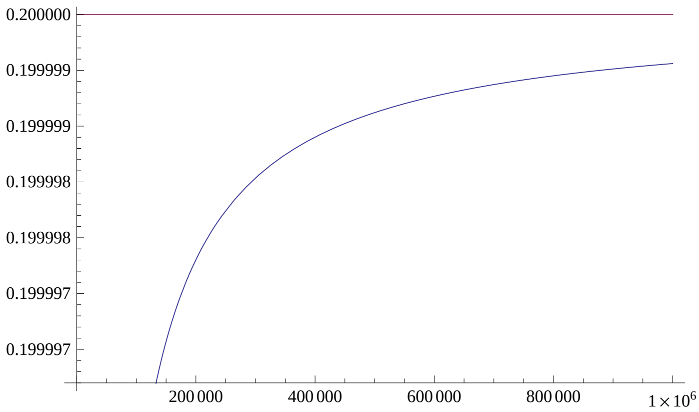
$$\int_a^b x^m dx \approx \sum_{k=0}^{n-1}(x_{k+1}-x_k)f(x_k) = \sum_{k=0}^{n-1}(ar^{k+1}-ar^k)(ar^k)^m = a^{m+1}(r-1)\sum_{k=0}^{n-1} r^{(m+1)k}$$
$$= a^{m+1}(r-1)\left\{\frac{1-r^{(m+1)n}}{1-r^{m+1}}\right\} = a^{m+1}\left\{(\frac{b}{a})^{\frac{1}{n}}-1\right\}\left\{\frac{1-(\frac{b}{a})^{m+1}}{1-(\frac{b}{a})^{\frac{m+1}{n}}}\right\} \quad (3)$$

At this stage it is worth taking a breather and have a look at the last bit of (3). In it we can see how we can get $b^{m+1}-a^{m+1}$ but we have to get the factor $\frac{1}{m+1}$ which may seem a bit far fetched at this stage. However, with a bit more algebraic manipulation things will become a little clearer:

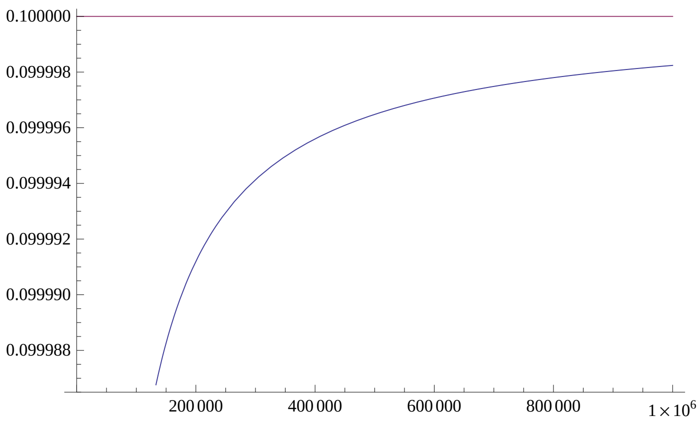
$$a^{m+1}\left\{(\frac{b}{a})^{\frac{1}{n}}-1\right\}\left\{\frac{1-(\frac{b}{a})^{m+1}}{1-(\frac{b}{a})^{\frac{m+1}{n}}}\right\} = \frac{a^{m+1}(b^{\frac{1}{n}}-a^{\frac{1}{n}})}{a^{\frac{1}{n}}a^{m+1}}\frac{(a^{m+1}-b^{m+1})}{(a^{\frac{m+1}{n}}-b^{\frac{m+1}{n}})}a^{\frac{m+1}{n}}$$
$$= a^{\frac{m}{n}}\left\{\frac{b^{\frac{1}{n}}-a^{\frac{1}{n}}}{b^{\frac{m+1}{n}}-a^{\frac{m+1}{n}}}\right\} \times (b^{m+1}-a^{m+1}) \quad (4)$$

In esssence we have to show that as $n \to \infty$, $a^{\frac{m}{n}}\left\{\frac{b^{\frac{1}{n}}-a^{\frac{1}{n}}}{b^{\frac{m+1}{n}}-a^{\frac{m+1}{n}}}\right\} \to \frac{1}{m+1}$. Since n is assumed large (so $\frac{1}{n}$ is small) why not try a binomial expansion of the terms involving b? We know that $b > a > 0$ so we can write $b = a+\delta$ where $\delta > 0$. Then $b^{\frac{1}{n}} - a^{\frac{1}{n}} = (a+\delta)^{\frac{1}{n}} - a^{\frac{1}{n}} > a^{\frac{1}{n}} + \frac{1}{n}a^{\frac{1}{n}-1}\,\delta - a^{\frac{1}{n}} = \frac{1}{n}a^{\frac{1}{n}-1}\delta$. Similarly, $b^{\frac{m+1}{n}} - a^{\frac{m+1}{n}} = (a+\delta)^{\frac{m+1}{n}} - a^{\frac{m+1}{n}} > a^{\frac{m+1}{n}} + \frac{m+1}{n}a^{\frac{m+1}{n}-1}\,\delta - a^{\frac{m+1}{n}} = \frac{m+1}{n}a^{\frac{m+1}{n}-1}\,\delta$. Accordingly, $\frac{1}{b^{\frac{m+1}{n}}-a^{\frac{m+1}{n}}} < \frac{n}{(m+1)\,\delta\, a^{\frac{m+1}{n}-1}}$. The ratio $a^{\frac{m}{n}}\left\{\frac{b^{\frac{1}{n}}-a^{\frac{1}{n}}}{b^{\frac{m+1}{n}}-a^{\frac{m+1}{n}}}\right\} < a^{\frac{m}{n}} \times \frac{1}{n}a^{\frac{1}{n}-1}\delta \times \frac{n}{(m+1)\,\delta\, a^{\frac{m+1}{n}-1}} = \frac{1}{m+1}$.

That this simple result involving m but neither a nor b emerges from the ratio $a^{\frac{m}{n}}\left\{\frac{b^{\frac{1}{n}}-a^{\frac{1}{n}}}{b^{\frac{m+1}{n}}-a^{\frac{m+1}{n}}}\right\}$, is rather amazing nothwithstanding it had to be that way if the theory made any sense. For those who harbour doubts, I offer two graphs produced using *Mathematica 8*. The horizontal line in each case is $\frac{1}{m+1}$.



$a = 1$
$b = 3$
$m = 4$
$1 \leq n \leq 10^6$



$a = 2$
$b = 100$
$m = 9$
$1 \leq n \leq 10^6$

3.1 Using self-similarity rather than Riemannian integration

It is possible to integrate $\int_0^1 x^n dx$, for instance, without employing the Riemannian approach. The following problem shows how.

The aim is to show that $\int_0^1 x^n dx = \frac{1}{n+1}$ by using the concept of self-similarity.

(1) Find an expression for $\int_0^1 x^n dx$ by splitting the domain of integration into $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ and making appropriate change of variables.
(2) Having found an expression for $\int_0^1 x^n dx$ develop an inductive argument using the binomial theorem to get the sought after result .

Solution

We start with $\int_0^1 x^n dx = \int_0^{\frac{1}{2}} x^n dx + \int_{\frac{1}{2}}^1 x^n dx$ and make the change of variable $u = 2x$ in the first integral and $u = 2(x - \frac{1}{2})$ in the second integral to get $\int_0^1 x^n dx = \frac{1}{2}\int_0^1 (\frac{x}{2})^n dx + \frac{1}{2}\int_0^1 (\frac{x}{2}+\frac{1}{2})^n dx$. When $n = 0$ we get $\int_0^1 dx = 1$ and when $n = 1$ we get:

$$\int_0^1 x\,dx = \frac{1}{2}\int_0^1 (\frac{x}{2})\,dx + \frac{1}{2}\int_0^1 (\frac{x}{2}+\frac{1}{2})\,dx = \frac{1}{2}\int_0^1 x\,dx + \frac{1}{4}\int_0^1 dx \tag{5}$$

(5) simplifies to:

$$\frac{1}{2}\int_0^1 x\,dx = \frac{1}{4} \quad \therefore \int_0^1 x\,dx = \frac{1}{2} \tag{6}$$

This establishes the formula $\frac{1}{n+1}$ for $n = 1$.

Following the same approach for $n = 2$ we get:

$$\int_0^1 x^2\,dx = \frac{1}{8}\int_0^1 x^2\,dx + \frac{1}{8}\int_0^1 x^2\,dx + \frac{1}{4}\int_0^1 x\,dx + \frac{1}{8}\int_0^1 dx \tag{7}$$

When the results for $n = 0, 1$ are used in (7) we get $\frac{3}{4}\int_0^1 x^2\,dx = \frac{1}{4}$ so that $\int_0^1 x^2\,dx = \frac{1}{3}$ as required. We now proceed for n generally:

$$\begin{aligned}\int_0^1 x^n dx &= \frac{1}{2}\int_0^1 (\frac{x}{2})^n dx + \frac{1}{2}\int_0^1 (\frac{x}{2}+\frac{1}{2})^n dx = \frac{1}{2^{n+1}}\int_0^1 x^n dx + \frac{1}{2^{n+1}}\int_0^1 (x+1)^n dx \\ &= \frac{1}{2^{n+1}}\Big\{\int_0^1 x^n dx + \int_0^1 \sum_{k=0}^{n}\binom{n}{k} x^k dx\Big\} = \frac{1}{2^{n+1}}\Big\{2\int_0^1 x^n dx + \sum_{k=0}^{n-1}\binom{n}{k}\int_0^1 x^k dx\Big\} \\ &= \frac{1}{2^{n+1}}\Big\{2\int_0^1 x^n\,dx + \sum_{k=0}^{n-1}\binom{n}{k}\frac{1}{k+1}\Big\}\end{aligned} \tag{8}$$

Where we have employed a standard induction in the last line of (8). Thus we get:

$$\left(\frac{2^n-1}{2^n}\right)\int_0^1 x^n\,dx = \frac{1}{2^{n+1}}\left\{\sum_{k=0}^{n-1}\binom{n}{k}\frac{1}{k+1}\right\} \tag{9}$$

To simplify (9) simply note that $\frac{n+1}{k+1}\binom{n}{k} = \frac{(n+1)!}{(n-k)!\,(k+1)!} = \binom{n+1}{k+1}$ thus:

$$\begin{aligned}(n+1)\sum_{k=0}^{n-1}\binom{n}{k}\frac{1}{k+1} &= \sum_{k=0}^{n-1}\binom{n+1}{k+1} = \sum_{j=1}^{n}\binom{n+1}{j} \text{ where} j = k+1\\ &= \sum_{j=0}^{n+1}\binom{n+1}{j} - 2 = 2^{n+1}-2 \quad \text{after adjusting for the terms } j = 0, n+1\end{aligned} \tag{10}$$

Therefore (10) becomes:

$$\sum_{k=0}^{n-1}\binom{n}{k}\frac{1}{k+1} = \frac{1}{n+1}\,(2^{n+1}-2) \tag{11}$$

So (9) becomes:

$$\left(\frac{2^n-1}{2^n}\right)\int_0^1 x^n\,dx = \frac{1}{2^{n+1}}\,\frac{(2^{n+1}-2)}{n+1} \tag{12}$$

Finally, (12) gives the desired result:

$$\int_0^1 x^n\,dx = \frac{1}{n+1} \tag{13}$$

This technique does not work more generally, however, it does have applications in the integration of fractals.

4 Example involving a trigonometrical integrand

In this example we use our bare hands to integrate $\int_a^b \cos mx dx$. We know the answer is $\frac{\sin mb - \sin ma}{m}$. As in the simple example we use equal subdivisions of [a,b] as follows: $x_k = a + k\frac{(b-a)}{n}$ for $k = 0, \dots, n$. In what follows let $u = \frac{m(b-a)}{n}$

$$\begin{aligned}\int_a^b \cos mx dx &\approx \sum_{k=0}^{n-1}(x_{k+1} - x_k)f(x_k) = \sum_{k=0}^{n-1} \frac{(b-a)}{n} \cos\big[ma + \frac{mk(b-a)}{n}\big] \\ &= \frac{u}{m}\sum_{k=0}^{n-1} \cos\big(ma + ku\big) = \frac{u}{m}\sum_{k=0}^{n-1}\big\{\cos ma \cos ku - \sin ma \sin ku\big\} \\ &= \frac{u}{m}\Bigg\{\cos ma \sum_{k=0}^{n-1}\cos ku - \sin ma \sum_{k=0}^{n-1} \sin ku\Bigg\} \quad (14)\end{aligned}$$

At this stage we need expressions for $\sum_{k=0}^{n-1}\cos ku$ and $\sum_{k=0}^{n-1}\sin ku$. If you look at my article on the Dirichlet kernel `http://www.gotohaggstrom.com/The%20good,%20the%20bad,%20and%20the%20ugly%20of%20kernels.pdf` you will find that:

$$\sum_{k=1}^{n}\cos ku = \frac{1}{2}\Bigg\{-1 + \frac{\sin((n+\frac{1}{2})u)}{\sin(\frac{u}{2})}\Bigg\} \quad (15)$$

The best way to prove (6) is not via induction (as indicated in the kernel paper this is monumentally tedious and error prone) but via a telescoping technique which I will set out below in relation to $\sum_{k=0}^{n-1}\sin ku$.

Adjusting for the fact that k runs from 0 we have to add 1 to (6) and so we get:

$$\sum_{k=0}^{n}\cos ku = \frac{1}{2}\Bigg\{1 + \frac{\sin((n+\frac{1}{2})u)}{\sin(\frac{u}{2})}\Bigg\} \quad (16)$$

To find a nice expression for $\sum_{k=1}^{n}\sin ku$ and hence the sought after sum, the best way is to use a telescoping technique as follows.

$\cos(ku - \frac{u}{2}) = \cos ku \cos(\frac{u}{2}) + \sin ku \sin(\frac{u}{2})$
$\cos(ku + \frac{u}{2}) = \cos ku \cos(\frac{u}{2}) - \sin ku \sin(\frac{u}{2})$

Therefore, $\sin(\frac{u}{2})\ \sin ku = \frac{1}{2}\Bigg\{\ \cos(ku - \frac{u}{2}) - \cos(ku + \frac{u}{2})\Bigg\}$. Summing from 1 to n we get cancellations and are left with the first and last terms:

$$\sin(\frac{u}{2})\sum_{k=1}^{n}\sin ku = \frac{1}{2}\sum_{k=1}^{n}\Bigg\{\cos(ku-\frac{u}{2})-\cos(ku+\frac{u}{2})\Bigg\} = \frac{1}{2}\sum_{k=1}^{n}\Bigg\{\cos(\frac{u}{2})-\cos(\frac{3u}{2})+\cos(\frac{3u}{2})-\cos(\frac{5u}{2})$$
$$+\cdots+\cos\Bigg((n-1)u-\frac{u}{2}\Bigg)-\cos\Bigg((n-1)u+\frac{u}{2}\Bigg)+\cos\Bigg(nu-\frac{u}{2}\Bigg)-\cos\Bigg(nu+\frac{u}{2}\Bigg)\Bigg\}$$
$$=\frac{1}{2}\Bigg\{\cos(\frac{u}{2})-\Bigg[\cos nu\ \cos(\frac{u}{2})-\sin nu\sin(\frac{u}{2})\Bigg]\Bigg\} = \frac{1}{2}\Bigg\{\cos(\frac{u}{2})\Bigg[1-\cos nu\Bigg]+\sin nu\sin(\frac{u}{2})\Bigg\}$$
$$=\frac{1}{2}\Bigg\{\cos(\frac{u}{2})2\sin^2(\frac{nu}{2})+\sin nu\sin(\frac{u}{2})\Bigg\} = \frac{1}{2}\Bigg\{\cos(\frac{u}{2})2\sin^2(\frac{nu}{2})+2\sin(\frac{nu}{2})\ \cos(\frac{nu}{2})\sin(\frac{u}{2})\Bigg\}$$
$$=\sin(\frac{nu}{2})\Bigg\{\cos(\frac{u}{2})\ \sin(\frac{nu}{2})+\sin(\frac{nu}{2})\ \cos(\frac{nu}{2})\sin(\frac{u}{2})\Bigg\} = \sin(\frac{nu}{2})\ \sin\Bigg(\frac{(n+1)u}{2}\Bigg) \tag{17}$$

Therefore,

$$\sum_{k=1}^{n}\sin ku = \frac{\sin\Big(\frac{(n+1)u}{2}\Big)\ \sin(\frac{nu}{2})}{\sin(\frac{u}{2})} \tag{18}$$

Using (9) it follows that:

$$\sum_{k=0}^{n-1}\sin ku = \frac{\sin(\frac{nu}{2})\ \sin\Big(\frac{(n-1)u}{2}\Big)}{\sin(\frac{u}{2})} \tag{19}$$

Going back to (5) we have:

$$\int_a^b \cos mx dx \approx \frac{u}{m}\Big\{\cos ma\sum_{k=0}^{n-1}\cos ku - \sin ma\sum_{k=0}^{n-1}\sin ku\Big\} = A + B \tag{20}$$

where $A = \frac{u}{m}\cos ma\Big\{\frac{1}{2}+\frac{1}{2}\frac{\sin((n+\frac{1}{2})u)}{\sin(\frac{u}{2})}\Big\}$ and $B = -\frac{u}{m}\ \sin ma\ \frac{\sin(\frac{nu}{2})\ \sin\left(\frac{(n-1)u}{2}\right)}{\sin(\frac{u}{2})}$

As a first observation in relation to A, since $u = \frac{m(b-a)}{n}$, $\frac{u}{m}\cos ma \to 0$ as $n \to \infty$ so we can ignore it from the further analysis. In both A and B there is the term $\frac{\frac{u}{2}}{\sin(\frac{u}{2})}$ which approaches 1 as $n \to \infty$. To see this just recall that $\frac{\sin x}{x} \to 1$ as $x \to 0$ as does its reciprocal. Thus we can ignore this factor in A and B when looking at the limiting behaviour.

Thus A boils down to (after adjusting for $\frac{\frac{u}{2}}{\sin(\frac{u}{2})}$):

$$\frac{\cos ma}{m}\ \sin\left[\left(n+\frac{1}{2}\right)\frac{m(b-a)}{n}\right] = \frac{\cos ma}{m}\ \sin\left[\Big(mb-ma)\Big)\left(1+\frac{1}{2n}\right)\right]$$
$$= \frac{\cos ma}{m}\left\{\sin\left(mb(1+\frac{1}{2n})\right)\ \cos\left(ma(1+\frac{1}{2n})\right) - \sin\left(ma(1+\frac{1}{2n})\right)\ \cos\left(mb(1+\frac{1}{2n})\right)\right\} \tag{21}$$

Owing to the continuity of $\sin x$ and $\cos x$, it follows that as $n \to \infty$, the final line of (12) approaches:

$$\frac{\cos ma}{m}\left\{\sin(mb)\ \cos(ma) - \sin(ma)\ \cos(mb)\right\} = \frac{\cos ma\ \sin(mb-ma)}{m} \tag{22}$$

We adjust B as follows by a factor of 2:

$B = -\frac{u}{m}\sin ma \frac{\sin(\frac{nu}{2})\,\sin\left(\frac{(n-1)u}{2}\right)}{\sin(\frac{u}{2})} = -\frac{2}{m}\frac{\frac{u}{2}}{\sin(\frac{u}{2})}\sin ma\ \sin(\frac{nu}{2})\ \sin\left(\frac{(n-1)u}{2}\right)$ so after ignoring the factor $\frac{\frac{u}{2}}{\sin(\frac{u}{2})}$ as discussed above we have B in the following form:

$$
\begin{aligned}
B &= \frac{-2}{m}\sin ma\ \sin(\frac{nu}{2})\ \sin\left(\frac{(n-1)u}{2}\right) = \frac{-2}{m}\sin ma\ \sin\left(\frac{mb-ma}{2}\right)\ \sin\left((mb{-}ma)(\frac{1}{2}{-}\frac{1}{2n})\right) \\
&= \frac{-2}{m}\sin ma\ \sin\left(\frac{mb-ma}{2}\right)\left\{\sin\left(mb(\frac{1}{2}{-}\frac{1}{2n})\right)\cos\left(ma(\frac{1}{2}{-}\frac{1}{2n})\right){-}\cos\left(mb(\frac{1}{2}{-}\frac{1}{2n})\right)\sin\left(ma(\frac{1}{2}{-}\frac{1}{2n})\right)\right\} \\
&\to \frac{-2}{m}\sin ma\ \sin\left(\frac{mb-ma}{2}\right)\left\{\sin\left(\frac{mb}{2}\right)\cos\left(\frac{ma}{2}\right)-\cos\left(\frac{mb}{2}\right)\sin\left(\frac{ma}{2}\right)\right\} \\
&= \frac{-2}{m}\sin ma\ \sin^2\left(\frac{mb-ma}{2}\right) = \frac{\sin ma}{m}\left\{\cos(mb-ma)-1\right\} \qquad (23)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
A+B &= \frac{\cos ma \sin(mb-ma)}{m} + \frac{\sin ma}{m}\left\{\cos(mb-ma)-1\right\} \\
&= \frac{\sin(mb-ma+ma)-\sin ma}{m} = \frac{\sin mb - \sin ma}{m} \qquad (24)
\end{aligned}
$$

Thus going back to (11) we see that $\int_a^b \cos mx dx = \frac{\sin mb - \sin ma}{m}$ as required.