Simplifying a matrix expression

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1 Introduction

Problem 1017 in the January 2015 College Mathematics Journal (Vol 46 No1) [1] gives the solution to the following problem. Let $\mathbf{d} = (d_1, d_2, d_3)^T \in \mathbb{R}^3$ be a vector where T denotes the transpose. Define the skew-symmetic matrix $\mathbf{T_d}$ as:

$$\mathbf{T_d} = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix} \tag{1}$$

Given the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, show that:

$$\mathbf{a}\mathbf{c}^T\mathbf{T_b} + \mathbf{b}\mathbf{a}^T\mathbf{T_c} + \mathbf{c}\mathbf{b}^T\mathbf{T_a} \tag{2}$$

can be written as a simple algebraic expression in the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

The solution by prolific and inventive French mathematical problem solver Michel Bataille is very slick and the purpose of this short note is to expand the proof to show the details and emphasise some basic facts about determinants. There are several typos in the published proof which are unfortunate.

2 Solution

The proof starts by asserting that $\mathbf{T_d}$ is an operator such that:

$$\mathbf{T_d}(\mathbf{x}) = \mathbf{d} \times \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^3 \tag{3}$$

That this is the case follows by performing the matrix multiplication and recalling the definition of a cross product in terms of its components ie

$$\mathbf{T_d}(\mathbf{x}) = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -d_3 x_2 + d_2 x_3 \\ d_3 x_1 - d_1 x_3 \\ -d_2 x_1 + d_1 x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{4}$$

Given the structure of (2) we can now see that for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$:

$$\mathbf{u}\mathbf{v}^T \mathbf{T_d}(\mathbf{x}) = \det(\mathbf{v}, \mathbf{d}, \mathbf{x})\, \mathbf{u} \tag{5}$$

Note that dimensions of the objects on the LHS give rise to a 3×1 column vector which is what the RHS is. We don't need any components to verify (5) because if we have a 3×3 matrix consisting of three column vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the determinant is:

$$\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \tag{6}$$

Thus, using (3) we have:

$$\mathbf{u}\mathbf{v}^T \mathbf{T_d}(\mathbf{x}) = \mathbf{u}\,(\mathbf{v}^T \mathbf{T_d}(\mathbf{x})\,) = \mathbf{u}\,(\mathbf{v}^T\, (\mathbf{d} \times \mathbf{x})\,) = \det(\mathbf{v}, \mathbf{d}, \mathbf{x})\, \mathbf{u} \tag{7}$$

If for all $\mathbf{x} \in \mathbb{R}^3$, we let:

$$L(\mathbf{x}) = \det(\mathbf{c}, \mathbf{b}, \mathbf{x})\, \mathbf{a} + \det(\mathbf{a}, \mathbf{c}, \mathbf{x})\, \mathbf{b} + \det(\mathbf{b}, \mathbf{a}, \mathbf{x})\, \mathbf{c} \tag{8}$$

then:

$$L(\mathbf{x}) = \mathbf{a}\mathbf{c}^T \mathbf{T_b}(\mathbf{x}) + \mathbf{b}\mathbf{a}^T \mathbf{T_c}(\mathbf{x}) + \mathbf{c}\mathbf{b}^T \mathbf{T_a}(\mathbf{x}) \tag{9}$$

Hence:

$$\begin{aligned} L(\mathbf{a}) &= \det(\mathbf{c}, \mathbf{b}, \mathbf{a})\, \mathbf{a} + \underbrace{\det(\mathbf{a}, \mathbf{c}, \mathbf{a})}_{=0}\, \mathbf{b} + \underbrace{\det(\mathbf{b}, \mathbf{a}, \mathbf{a})}_{=0}\, \mathbf{c} \\ L(\mathbf{b}) &= \underbrace{\det(\mathbf{c}, \mathbf{b}, \mathbf{b})}_{=0}\, \mathbf{a} + \det(\mathbf{a}, \mathbf{c}, \mathbf{b})\, \mathbf{b} + \underbrace{\det(\mathbf{b}, \mathbf{a}, \mathbf{b})}_{=0}\, \mathbf{c} \\ L(\mathbf{c}) &= \underbrace{\det(\mathbf{c}, \mathbf{b}, \mathbf{c})}_{=0}\, \mathbf{a} + \underbrace{\det(\mathbf{a}, \mathbf{c}, \mathbf{c})}_{=0}\, \mathbf{b} + \det(\mathbf{b}, \mathbf{a}, \mathbf{c})\, \mathbf{c} \\ \therefore L(\mathbf{a} + \mathbf{b} + \mathbf{c}) &= \det(\mathbf{c}, \mathbf{b}, \mathbf{a})\, \mathbf{a} + \det(\mathbf{a}, \mathbf{c}, \mathbf{b})\, \mathbf{b} + \det(\mathbf{b}, \mathbf{a}, \mathbf{c})\, \mathbf{c} \end{aligned} \tag{10}$$

Now:
$\det(\mathbf{c}, \mathbf{b}, \mathbf{a}) = -\det(\mathbf{c}, \mathbf{a}, \mathbf{b}) = \det(\mathbf{a}, \mathbf{c}, \mathbf{b}) = -\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$
$\det(\mathbf{a}, \mathbf{c}, \mathbf{b}) = -\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$
$\det(\mathbf{b}, \mathbf{a}, \mathbf{c}) = -\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$

Thus:

$$\begin{aligned} L(\mathbf{a} + \mathbf{b} + \mathbf{c}) &= -\det(\mathbf{a}, \mathbf{b}, \mathbf{c})\,(\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ \therefore L &= -\det(\mathbf{a}, \mathbf{b}, \mathbf{c})\, I_3 \end{aligned} \tag{11}$$

I_3 is the 3×3 identity matrix.

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are independent vectors then $\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq 0$. This is a consequence of a basic linear algebra result: $\mathbf{A}$ is non-singular ie $\mathbf{Ax} = \mathbf{0}$ only has a zero solution, or rank $\mathbf{A} = n$ or the rows/columns of $\mathbf{A}$ are linearly independent if and only if $\det(\mathbf{A}) \neq 0$.

Now if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are dependent vectors then $L(\mathbf{x}) = \mathbf{0}$ because $\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$. Thus in either case the relationship in (11) holds.

3 References

1. `http://www.maa.org/publications/periodicals/college-mathematics-journal/the-college-mathematics-journal`

4 History

Created 11/05/2015 28/09/2015 corrected typo