Solving a basic pursuit curve problem

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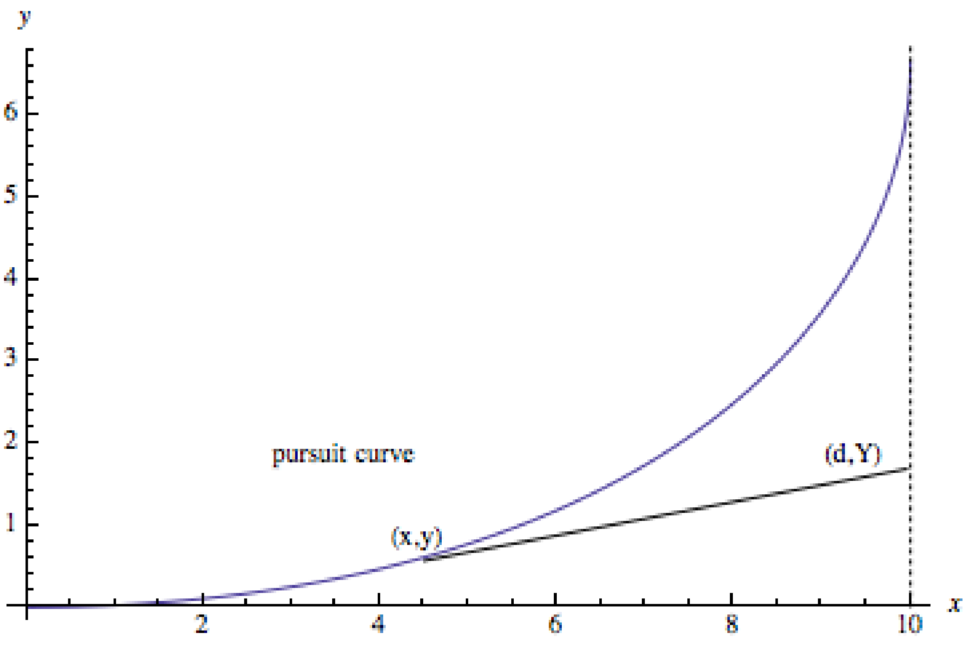
1 Background

Apparently the theory of pursuit curves was subjected to serious study in the 1700s as a result of the attraction of attacking ships laden with various forms of valuable goods (gold, spices etc). The classic problem involves a fast pirate ship which pursues a heavily laden treasure ship which tracks along a straight line. The ratio of the speeds of the ships is $r > 1$ (which is fixed) and the pirate captain spies the treasure ship initially at a distance d km away. Naturally the pirate captain tracks towards the treasure ship.



The treasure ship tracks along the line x = d and at any time when the pirate ship is at coordinates (x,y) the treasure ship is at coordinates (d,Y). Since the pirate ship is always tracking towards the treasure ship's coordinate the tangent from (x,y) must go through (d,Y). This gives the basic relationship:

$$\frac{dy}{dx} = \frac{Y - y}{d - x} \tag{1}$$

(1) is re-arranged as:

$$Y = (d - x)\frac{dy}{dx} + y \tag{2}$$

Now along the pursuit curve the arc length element is $ds = \sqrt{(dx)^2 + (dy)^2}$ so that the pirate ship's speed is:

$$\frac{ds}{dt} = r\left(\frac{dY}{dt}\right) \quad \text{since the ratio of the pirate ship's speed to the treasure ship is r} \tag{3}$$

Thus:

$$ds = r\,dY \quad \text{ie} \quad \sqrt{1 + (\frac{dy}{dx})^2} = r\,\frac{dY}{dx} \tag{4}$$

Now if we differentiate (2) with respect to x we get:

$$\frac{dY}{dx} = (d - x)\frac{d^2y}{dx^2} - \frac{dy}{dx} + \frac{dy}{dx} = (d - x)\frac{d^2y}{dx^2} \tag{5}$$

Using (4) we get the following nonlinear ordinary differential equation for the pursuit curve:

$$r(d - x)\frac{d^2y}{dx^2} = \sqrt{1 + (\frac{dy}{dx})^2} \tag{6}$$

A standard approach to solving (6) is to let $u = \frac{dy}{dx}$ and separating variables, giving:

$$\frac{rdu}{\sqrt{1 + u^2}} = \frac{dx}{d - x} \tag{7}$$

The integral of the left hand side of (7) is the inverse hyperbolic function $r\sinh^{-1}u$. If you look at integral tables for instance you will find that $\int \frac{r\,du}{\sqrt{1+u^2}}$ is sometimes given in its logarithmic form $\ln(u + \sqrt{1 + u^2})$, however, if you go down that route the solution will prove elusive. A much better approach is to go back to the basic definition of $\sinh^{-1} u$. Thus from (7) we have (note that rather than distinguishing constants at each intermediate step C will be a generic constant) :

$r\sinh^{-1} u = -\ln(d-x) + C$

$\sinh^{-1} u = \ln\left(\frac{C}{(d-x)^{\frac{1}{r}}}\right)$

Now let $\sinh^{-1} u = v$ so that $u = \sinh v = \frac{1}{2}(e^v - e^{-v})$

Then we have:

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{C}{(d-x)^{\frac{1}{r}}} - \frac{(d-x)^{\frac{1}{r}}}{C}\right) \tag{8}$$

C is found by noting that $\frac{dy}{dx} = 0$ when $x = 0$, thus:

$$0 = \left(\frac{C}{d^{\frac{1}{r}}} - \frac{d^{\frac{1}{r}}}{C}\right) \quad \therefore \quad C^2 = d^{\frac{2}{r}} \text{ So finally } C = d^{\frac{1}{r}} \tag{9}$$

Thus (8) becomes:

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{d^{\frac{1}{r}}}{(d-x)^{\frac{1}{r}}} - \frac{(d-x)^{\frac{1}{r}}}{d^{\frac{1}{r}}}\right) \tag{10}$$

One integrating (10) we get:

$$y = \frac{1}{2}\int\left(\frac{d^{\frac{1}{r}}}{(d-x)^{\frac{1}{r}}} - \frac{(d-x)^{\frac{1}{r}}}{d^{\frac{1}{r}}}\right)dx = \frac{1}{2}\left(\frac{-rd^{\frac{1}{r}}(d-x)^{1-\frac{1}{r}}}{r-1} + \frac{r(d-x)^{1+\frac{1}{r}}}{d^{\frac{1}{r}}(r+1)} + C'\right) \tag{11}$$

To evaluate C', note that when $x = 0$, $y = 0$, so:

$$0 = \frac{1}{2}\left(\frac{-rd^{\frac{1}{r}}d^{1-\frac{1}{r}}}{r-1} + \frac{rd^{1+\frac{1}{r}}}{d^{\frac{1}{r}}(r+1)} + C'\right) \quad \therefore C' = \frac{rd}{r-1} - \frac{rd}{r+1} = \frac{2rd}{r^2-1} \tag{12}$$

Substituting C' into (11) we get:

$$
\begin{aligned}
y &= \frac{1}{2}\left(\frac{-rd^{\frac{1}{r}}(d-x)^{1-\frac{1}{r}}}{r-1} + \frac{r(d-x)^{1+\frac{1}{r}}}{d^{\frac{1}{r}}(r+1)} + \frac{2rd}{r^2-1}\right) \\
&= \frac{1}{2}\left(\frac{-rd^{\frac{1}{r}}(1-\frac{x}{d})^{1-\frac{1}{r}}d^{1-\frac{1}{r}}}{r-1} + \frac{r(1-\frac{x}{d})^{1+\frac{1}{r}}\,d^{1+\frac{1}{r}}}{d^{\frac{1}{r}}(r+1)} + \frac{2rd}{r^2-1}\right) \\
&= \frac{1}{2}\left(\frac{-rd(1-\frac{x}{d})^{1-\frac{1}{r}}}{r-1} + \frac{rd(1-\frac{x}{d})^{1+\frac{1}{r}}}{r+1} + \frac{2rd}{r^2-1}\right) \\
&= \frac{rd}{2(r^2-1)}\left(2 + (r-1)(1-\frac{x}{d})^{1+\frac{1}{r}} - (r+1)(1-\frac{x}{d})^{1-\frac{1}{r}}\right) \qquad (13)
\end{aligned}
$$

It can be seen from (13) that the pirate ship will capture the treasure ship when $x = d$ which gives $y = \frac{rd}{r^2-1}$.

Vector techniques can be adpated to derive pursuit curve equations, for instance, see `http://mathworld.wolfram.com/PursuitCurve.html`

2 History

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29/03/2014: Typos in (1), (12) and (13) corrected thanks to email from Eugenio Pariani,Italy.