Special Relativity-how Einstein actually did it

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1 Introduction

Albert Einstein's June 1905 paper titled "On The Electrodynamics Of Moving Bodies" [1]] set out his explanation of relativistic principles. There is a vast body of literature on this paper and the purpose of this paper is not to deal with any controversies surrounding whether or not Einstein was aware of the Lorentz transformation at the time of his paper. Instead what I want to do is explain how Einstein actually "did the business", since the way he presented special relativity is simply not generally taught at undergraduate level. Indeed, he published various explanations of special relativity for different audiences, but what is interesting is how he framed the ideas in the beginning (see [2], [3]). If one reads his lectures given at Princeton in 1921 (see [3]) the development he gives of special relativity is quite different to that of the original 1905 paper. In [2] Einstein presents in Appendix 1 a derivation of the Lorentz transformation quite different to the 1905 paper. One author has noted that "Einstein himself never reverted to the arguments employed in Sec. 3 " of the 1905 paper [5, page 179] It is always instructive to go back to the originators of ideas to see how they fabricated their theoretical cloth. Feynman's path integral approach set out in his doctoral thesis is a classic example. Feynman's path integral idea caught on, but the same cannot be said for Einstein's approach in the 1905 paper.

In what follows I will be focusing solely on the kinematical part of Einstein's paper which gives rise to the Lorentz transformations. Readers can read Lorentz's original paper [4] to see where the differences exist.

2 The two postulates

Einstein starts by presenting two postulates in the following quotation [1, page1]:

“ Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the “light medium,” suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good.1 We will raise this conjecture (the purport of which will hereafter be called the “Principle of Relativity”) to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell’s theory for stationary bodies.”

I will refer to these two postulates as the “relativity principle” and the “velocity independence principle” respectively.

The velocity independence principle is, at first blush, difficult to accept since if light is travelling towards me at c and I run away from it at velocity v I ought to measure the velocity of the ray as $c-v$. Similarly if I approach the oncoming light ray at velocity v I should measure its velocity as $c+v$. That this is not the case is the crux of Einstein’s paper and involves a search for a functional relationship that obeys both postulates and involves an abandonment of the fixation of time in all reference frames.

Einstein gives the following explanation: “ Let a ray of light depart from A at the time (i.e. the time in the stationary system) t_A , let it be reflected at B at the time t_B , and reach A again at the time $t_{A'}$. Taking into consideration the principle of the constancy of the velocity of light we find that:

$t_B - t_A = \frac{r_{AB}}{c-v}$ and $t_{A'} - t_B = \frac{r_{AB}}{c+v}$

where r_{AB} denotes the length of the moving rod—measured in the stationary system. Observers moving with the moving rod would thus find that the two clocks were not synchronous, while observers in the stationary system would declare the clocks to be synchronous. “

3 Simultaneity

Einstein lays the foundation for the concept of simultaneity as follows:

" If at the point A of space there is a clock, an observer at A can determine the time values of events in the immediate proximity of A by finding the positions of the hands which are simultaneous with these events. If there is at the point B of space another clock in all respects resembling the one at A, it is possible for an observer at B to determine the time values of events in the immediate neighbourhood of B. But it is not possible without further assumption to compare, in respect of time, an event at A with an event at B. We have so far defined only an "A time" and a "B time." We have not defined a common "time" for A and B, for the latter cannot be defined at all unless we establish by definition that the "time" required by light to travel from A to B equals the "time" it requires to travel from B to A. Let a ray of light start at the "A time" t_A from A towards B, let it at the "B time" t_B be reflected at B in the direction of A, and arrive again at A at the "A time" $t_{A'}$. In accordance with definition the two clocks synchronize if

$$t_B - t_A = t_{A'} - t_B \tag{1}$$

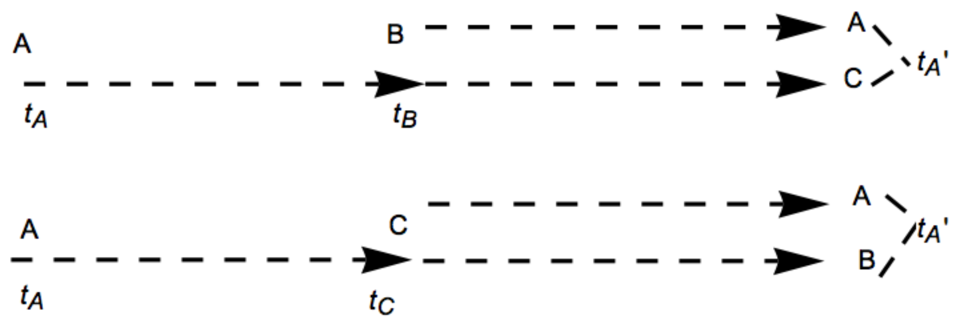
We assume that this definition of synchronism is free from contradictions, and possible for any number of points; and that the following relations are universally valid:— 1. If the clock at B synchronizes with the clock at A, the clock at A synchronizes with the clock at B. 2. If the clock at A synchronizes with the clock at B and also with the clock at C, the clocks at B and C also synchronize with each other. Thus with the help of certain imaginary physical experiments we have settled what is to be understood by synchronous stationary clocks located at different places, and have evidently obtained a definition of "simultaneous," or "synchronous," and of "time." The "time" of an event is that which is given simultaneously with the event by a stationary clock located at the place of the event, this clock being synchronous, and indeed synchronous for all time determinations, with a specified stationary clock. In agreement with experience we further assume the quantity

$$\frac{2AB}{t_{A'} - t_A} = c \tag{2}$$

to be a universal constant—the velocity of light in empty space. It is essential to have time defined by means of stationary clocks in the stationary system, and the time now defined being appropriate to the stationary system we call it "the time of the stationary system."

Einstein's concept of simultaneity is thus symmetric and transitive, however, he gives no proof. Symmetry is easy to accept for if A synchronizes with B (written as $A \sim B$ the two time differences in (1) should be the same and so B synchronizes with A ($B \sim A$). One could arrive at the same result by issuing a light ray from a central point to two equally spaced reflectors. Einstein's synchronization concept depends on the constancy of the velocity of light.

The relation that if $A \sim B$ and $A \sim C$ then $B \sim A$ is actually transitivity in disguise since we have by the symmetry relation $A \sim B$ implies $B \sim A$ and so with $A \sim C$ it ought to follow that $B \sim C$. Although Einstein presents this as almost an axiomatic property, is it derivable in a consistent way from his definition of simultaneity?



The diagram above represents (the top figure) light travelling from A to B and being reflected back to A and then being reflected to C. The bottom figure represents light travelling from A to C and then being reflected back to A and being reflected to B. Thus between the diagrams we capture light being reflected between A and B, A and C and B and C. Note that lengths of all arrows are equal and by the constancy of the velocity of light, the time intervals must all be equal. Thus B is indeed synchronised with C.

4 How Einstein did the transformation of coordinates

The path Einstein followed in demonstrating the coordinate transformation between the moving and stationary coordinate systems is not taught as far as I can tell, or if it is, it is probably viewed as idiosyncratic or a curiosity. Indeed, he never replicated the logic of the 1905 paper himself. What he did is this. The coordinates of the stationary system are x, y, z, t and of the moving system ξ, η, ζ, τ. The moving system moves with constant velocity v with respect to the stationary system. Assuming the homogeneity of space and time, Einstein is looking for a linear relationship. If we set $x^{'} = x - vt$ then a point at rest in the moving system must have a system of values of $x^{'}, y, z$ independent of time. Time in the moving system, τ , is defined in terms of $x^{'}, y, z$ and t.

He imagines a moving system of coordinates in which a light ray is emitted at time τ_0 from the origin along the x axis to $x^{'}$ at time τ_1 and reflected back to the origin of the coordinate system at time τ_2. It follows that

$$\tau_1 = \frac{\tau_0 + \tau_2}{2} \tag{3}$$

This can derived from the basic synchronization relationship:

$\tau_1 - \tau_0 = \tau_2 - \tau_1$

The constancy of the speed of light is implicitly assumed in deriving (3).

Next he inserts the arguments of the function τ and applies the constancy of light in the stationary system to get:

$$\frac{1}{2}\Big[\tau(0,0,0,t) + \tau\Big(0,0,0,t + \frac{x^{'}}{c-v} + \frac{x^{'}}{c+v}\Big)\Big] = \tau\Big(x^{'},0,0,t + \frac{x^{'}}{c-v}\Big) \tag{4}$$

The next step is to assume that $x^{'}$ is infinitesimal and hence the following partial differential equation emerges:

$$\frac{1}{2}\Big(\frac{1}{c-v} + \frac{1}{c+v}\Big)\frac{\partial \tau}{\partial t} = \frac{\partial \tau}{\partial x'} + \frac{1}{c-v}\frac{\partial \tau}{\partial t} \tag{5}$$

When you look at the standard undergraduate introductions to special relativity you do not find this partial differential equation (PDE) which looks suspiciously like the transport equation (with c a constant):

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{6}$$

subject to $u(x,0) = f(x)$.

Let's flesh out the derivation of the PDE - from (4):

$$\frac{1}{2}\Big[\tau(0,0,0,t)+\tau\Big(0,0,0,t+\frac{x^{'}}{c-v}+\frac{x^{'}}{c+v}\Big)-\tau(0,0,0,t)\Big]=\tau\Big(x^{'},0,0,t+\frac{x^{'}}{c-v}\Big)-\frac{1}{2}\tau(0,0,0,t)$$
$$\frac{1}{2}\Big[\tau\Big(0,0,0,t+\frac{x^{'}}{c-v}+\frac{x^{'}}{c+v}\Big)-\tau(0,0,0,t)\Big]=\tau\Big(x^{'},0,0,t+\frac{x^{'}}{c-v}\Big)-\tau(0,0,0,t) \quad (7)$$

Therefore:

$$\frac{1}{2}\Big[\tau\Big(0,0,0,t+\frac{x^{'}}{c-v}+\frac{x^{'}}{c+v}\Big)-\tau(0,0,0,t)\Big]=\underbrace{\tau\Big(x^{'},0,0,t+\frac{x^{'}}{c-v}\Big)-\tau\Big(0,0,0,t+\frac{x^{'}}{c-v}\Big)}_{\text{in the limit } \frac{\partial\tau}{\partial x^{'}}}$$
$$+\underbrace{\tau\Big(0,0,0,t+\frac{x^{'}}{c-v}\Big)-\tau(0,0,0,t)}_{\text{in the limit } \frac{1}{c-v}\frac{\partial\tau}{\partial t}} \quad (8)$$

Now since v is constant and the velocity of light c is assumed constant in the stationary frame and $x^{'}$ is infinitesimal, we have that $\frac{x^{'}}{c-v}+\frac{x^{'}}{c+v}$ has the dimensions of time. But if we treat c, v as dimensionless constants and employ a velocity unit vector "1" we can write $\frac{x^{'}}{c-v}+\frac{x^{'}}{c+v}=(\frac{1}{c-v}+\frac{1}{c+v})\frac{x^{'}}{1}=(\frac{1}{c-v}+\frac{1}{c+v})\,\Delta t$ which still has the dimensions of time (infinitesimal $x^{'}$ is divided by a unit velocity). By doing this we can use the assumed linearity of τ to get the constants "out the front". Thus:

$$\frac{1}{2}\Big[\tau\Big(0,0,0,t+\frac{x^{'}}{c-v}+\frac{x^{'}}{c+v}\Big)-\tau(0,0,0,t)\Big]=\frac{1}{2}\Big[\tau\Big(0,0,0,t+(\frac{1}{c-v}+\frac{1}{c+v})\Delta t\Big)-\tau(0,0,0,t)\Big] \quad (9)$$

As $\Delta t\to 0$ (ie as $x^{'}\to 0$) the RHS of (9) approaches $\frac{1}{2}(\frac{1}{c-v}+\frac{1}{c+v})\,\frac{\partial\tau}{\partial t}$. So the LHS of (4) is:

$$\frac{1}{2}(\frac{1}{c-v}+\frac{1}{c+v})\,\frac{\partial\tau}{\partial t} \quad (10)$$

Going back to the RHS of (8), in the limit it becomes:

$$\frac{\partial \tau}{\partial x'} + \frac{1}{c-v}\frac{\partial \tau}{\partial t} \tag{11}$$

Thus we do get (5) which simplifies to:

$$\frac{\partial \tau}{\partial x'} + \frac{v}{c^2-v^2}\frac{\partial \tau}{\partial t} = 0 \tag{12}$$

Equation (12) is valid for all $x^{'}, y, z$ since the origin is arbitrary. When viewed from the stationary system the light travelling in the Y, Z directions of the moving system with velocity $\sqrt{c^2-v^2}$ (just draw a right-anged triangle representing the velocities). On this basis, we must then have:

$$\begin{aligned} \frac{\partial \tau}{\partial y} &= 0 \\ \frac{\partial \tau}{\partial z} &= 0 \end{aligned} \tag{13}$$

Since τ is linear, Einstein asserts the following functional relationship:

$$\tau = a\Big(t - \frac{v}{c^2-v^2}x^{'}\Big) \tag{14}$$

where a is a function $\phi(v)$ which is to be determined , and it is assumed that at the origin of the moving system $\tau = 0$ when $t = 0$.

This is verified as follows. By linearity we have that:

$$\tau = a\Big(t + bx^{'}\Big) \tag{15}$$

b a constant

But:

$$\begin{aligned} \frac{\partial \tau}{\partial x'} &= ab \\ \frac{\partial \tau}{\partial t} &= a \end{aligned} \tag{16}$$

Therefore, using (12):

$$\frac{\partial\tau}{\partial x'} + \frac{v}{c^2 - v^2}\frac{\partial\tau}{\partial t} = ab + \frac{v}{c^2 - v^2}a = 0$$
$$\therefore b = \frac{-v}{c^2 - v^2} \tag{17}$$

Hence (14) follows.

The next step is to get expressions for ξ, η, ζ by applying the velocity independence principle in combination with the relativity principle. So for a light ray emitted at time $\tau = 0$ in the direction of increasing ξ we have:

$$\xi = c\tau = ac\left(t - \frac{v}{c^2 - v^2}x'\right) \tag{18}$$

But the ray moves relatively to the initial point of the moving system when measured in the stationary system with velocity $c - v$ so we have:

$$\frac{x'}{c - v} = t \tag{19}$$

Substituting $\frac{x'}{c-v} = t$ into (18) we get:

$$\begin{aligned} \xi =& ac\left(t - \frac{v}{c^2 - v^2}x'\right) \\ =& ac\Big(\frac{x'}{c - v} - \frac{vx'}{c^2 - v^2}\Big) \\ =& ac\frac{\left(x'(c + v) - vx'\right)}{c^2 - v^2} \\ =& \frac{ac^2x'}{c^2 - v^2} \end{aligned} \tag{20}$$

For η and ζ:

$$\eta = c\tau = ac\left(t - \frac{v}{c^2 - v^2}x'\right) \tag{21}$$

but when $\frac{y}{\sqrt{c^2 - v^2}} = t$, $x' = 0$.

Therefore:

$$\eta = c\tau = act = \frac{acy}{\sqrt{c^2 - v^2}} \tag{22}$$

Similarly:

$$\zeta = c\tau = act = \frac{acz}{\sqrt{c^2 - v^2}} \tag{23}$$

Einstein then says that "by substituting for $x^{'}$ its value we obtain:

$$\tau = \phi(v)\beta(t - \frac{vx}{c^2}) \tag{24}$$

$$\xi = \phi(v)\beta(x - vt) \tag{25}$$

$$\eta = \phi(v)y \tag{26}$$

$$\zeta = \phi(v)z \tag{27}$$

where $\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ "

This step is confusing at first sight since when one substitutes $x^{'} = x - vt$ into (14) you actually get this:

$$\begin{aligned} \tau =& a\left(t - \frac{v}{c^2 - v^2}(x - vt)\right) \\ =& a\left(\frac{c^2 t - vx}{c^2 - v^2}\right) \\ =& \frac{ac^2}{c^2 - v^2}\left(t - \frac{vx}{c^2}\right) \\ =& a\frac{1}{1 - \frac{v^2}{c^2}}\left(t - \frac{vx}{c^2}\right) \\ =& a\beta^2\left(t - \frac{vx}{c^2}\right) \end{aligned} \tag{28}$$

The factor β^2 instead of β is just sloppy notation and Einstein has effectively assumed that $a\beta = \phi(v)$ and when this is done you do get (24). Ultimately he establishes that $\phi(v) = 1$ so $a = \frac{1}{\beta}$.

Einstein's next step is to prove that any ray of light, as measured in the moving system, is propagated with velocity c so that the principle of velocity independence is compatible with the principle of relativity.

He assumes that at time $t = \tau = 0$ when the origin of the coordinates is common to both the stationary and moving systems, a spherical wave is emitted with velocity c in the stationary system. Thus if (x, y, z) is a point obtained by this wave we must have:

$$x^2 + y^2 + z^2 = c^2 t^2 \tag{29}$$

He then says that the transformed equation is as follows :

$$\xi^2 + \eta^2 + \zeta^2 = c^2 \tau^2 \tag{30}$$

Let us write the transformation equations (24)-(27) in the following format:

$$\tau = \phi(v)\beta\, t - \frac{\phi(v)\beta v}{c^2}\, x + 0y + 0z \tag{31}$$

$$\xi = -\phi(v)\beta\, v\, t + \phi(v)\beta\, x + 0y + 0z \tag{32}$$

$$\eta = 0t + 0x + \phi(v)y + 0z \tag{33}$$

$$\zeta = 0t + 0x + 0y + \phi(v)z \tag{34}$$

(32) $\times \frac{v}{c^2}$ + (31) becomes:

$$\begin{aligned} \frac{\xi v}{c^2} + \tau =& \phi(v)\beta\, t - \frac{\phi(v)\beta v^2\, t}{c^2} \\ =& \phi(v)\beta t \Big(1 - \frac{v^2}{c^2}\Big) \\ =& \phi(v)\beta t\, \frac{1}{\beta^2} \\ =& \frac{\phi(v) t}{\beta} \\ \therefore t =& \frac{\beta}{\phi(v)}\Big(\tau + \frac{\xi v}{c^2}\Big) \end{aligned} \tag{35}$$

We also have that (31) $\times v +$ (32) becomes:

$$
\begin{aligned}
\xi + v\tau =& \phi(v)\beta x - \frac{\phi(v)\beta v^2 x}{c^2} \\
=& \phi(v)\beta\Big(1 - \frac{v^2}{c^2}\Big)\, x \\
=& \phi(v)\beta x\, \frac{1}{\beta^2} \\
=& \frac{\phi(v)x}{\beta} \\
\therefore x =& \frac{\beta}{\phi(v)}\Big(\xi + v\tau\Big)
\end{aligned} \tag{36}
$$

We also have:

$$
y = \frac{\eta}{\phi(v)} \tag{37}
$$

$$
z = \frac{\zeta}{\phi(v)} \tag{38}
$$

Now we simply substitute x, y, z into (29):

$$
\begin{aligned}
0 =& x^2 + y^2 + z^2 - c^2t^2 \\
=& \frac{\beta^2}{\phi(v)^2}\big(\xi^2 + 2\xi v\tau + v^2\tau^2\big) + \Big(\frac{\eta}{\phi(v)}\Big)^2 + \Big(\frac{\zeta}{\phi(v)}\Big)^2 - \frac{c^2\beta^2}{\phi(v)^2c^4}\Big(\xi^2v^2 + 2\xi vc^2\tau + c^4\tau^2\Big) \\
=& \frac{\beta^2}{\phi(v)^2}\Big(1 - \frac{v^2}{c^2}\Big)\xi^2 + \frac{2\beta^2\xi v\tau}{\phi(v)^2} + \frac{\beta^2v^2\tau^2}{\phi(v)^2} - \frac{2\beta^2\xi v\tau}{\phi(v)^2} - \frac{\beta^2c^2\tau^2}{\phi(v)^2} + \Big(\frac{\eta}{\phi(v)}\Big)^2 + \Big(\frac{\zeta}{\phi(v)}\Big)^2 \\
=& \frac{\xi^2}{\phi(v)^2} + \frac{\beta^2(v^2 - c^2)\tau^2}{\phi(v)^2} + \Big(\frac{\eta}{\phi(v)}\Big)^2 + \Big(\frac{\zeta}{\phi(v)}\Big)^2 \\
=& \frac{\xi^2}{\phi(v)^2} + \frac{c^2}{c^2 - v^2}\frac{v^2 - c^2}{\phi(v)^2}\,\tau^2 + \Big(\frac{\eta}{\phi(v)}\Big)^2 + \Big(\frac{\zeta}{\phi(v)}\Big)^2 \\
&\therefore \xi^2 + \eta^2 + \zeta^2 = c^2\tau^2
\end{aligned} \tag{39}
$$

The temptation to write (31)-(34) in terms of a matrix and then invert the matrix should be avoided - it is actually more complicated although it does potentially illuminate other things such as orthogonality.

Einstein observes that the "wave under consideration is therefore no less a spherical wave with velocity of propagation c when viewed in the moving system " [1, page 8]. Thus the two fundamental postulates are compatible.

Einstein's next step is to determine $\phi(v)$. To do this he introduces another coordinate system $K^{'}$ which "relatively to the system k is in a state of parallel translatory motion parallel to the axis of Ξ, such that the origin of coordinates of system $K^{'}$ moves with velocity $-v$ on the axis of Ξ. At time $t = 0$ let all three origins coincide, and when $t = x = y = z = 0$ let the time $t^{'}$ of the system $K^{'}$ be zero ". Note the footnote on page 8 of [1] which clarifies an ambiguity in the coordinate naming.

To recapitulate, Einstein now has 3 systems of coordinates:

System k which is moving in the direction of increasing x of stationary system K. System K has spatial coordinates x, y, z while moving system k has spatial coordinates ξ, η, ζ. Finally, there is the system $K^{'}$ which moves relatively to system k with velocity $-v$.

Einstein explains the next step as a "twofold application of our equations of transformation" which result in the following relationships:

$$t^{'} = \phi(-v)\beta(-v)(\tau + \frac{v\xi}{c^2}) = \phi(v)\phi(-v)\,t \tag{40}$$

$$x^{'} = \phi(-v)\beta(-v)(\xi + v\tau) = \phi(v)\phi(-v)\,x \tag{41}$$

$$y^{'} = \phi(-v)\eta = \phi(v)\phi(-v)\,y \tag{42}$$

$$z^{'} = \phi(-v)\zeta = \phi(v)\phi(-v)\,z \tag{43}$$

What Einstein has done here appears to be as follows. First, in relation to (40) we need to recall (35) ie:

$$t = \frac{\beta(v)}{\phi(v)}\Big(\frac{\xi v}{c^2} + \tau\Big) \tag{44}$$

where we have explicity identified the dependence of β on v since $\beta(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \beta(-v)$.

From (24) with $t \rightarrow \tau$, $x \rightarrow \xi$ and $v \rightarrow -v$ we have:

$$t^{'} = \phi(-v)\beta(-v)\Big(\tau + \frac{\xi v}{c^2}\Big) \tag{45}$$

But from (35):

$$t = \frac{\beta(v)}{\phi(v)}\Big(\tau + \frac{\xi v}{c^2}\Big) \tag{46}$$

Therefore:

$$\begin{aligned} \phi(v)\phi(-v)t =& \phi(-v)\beta(v)\Big(\tau + \frac{\xi v}{c^2}\Big) \\ =& \phi(-v)\beta(-v)\Big(\tau + \frac{\xi v}{c^2}\Big) \\ =& t^{'} \end{aligned} \tag{47}$$

Similarly, from (25) with $t \rightarrow \tau$, $x \rightarrow \xi$ and $v \rightarrow -v$, we have:

$$x^{'} = \phi(-v)\beta(-v)\Big(\xi + v\tau\Big) \tag{48}$$

But from (36) we have:

$$\begin{aligned} x =& \frac{\beta(v)}{\phi(v)}\Big(\xi + v\tau\Big) \\ =& \frac{\beta(-v)}{\phi(v)}\Big(\xi + v\tau\Big) \end{aligned} \tag{49}$$

Therefore:

$$\phi(-v)\phi(v)x = \phi(-v)\beta(-v)\Big(\xi + v\tau\Big) = x^{'} \tag{50}$$

From (26) we have:

$$y^{'} = \phi(-v)\eta \tag{51}$$

But $y = \frac{\eta}{\phi(v)}$ therefore:

$$y^{'} = \phi(-v)\phi(v)y \tag{52}$$

Finally, from (27) we have:

$$z^{'} = \phi(-v)\zeta \tag{53}$$

But $z = \frac{\zeta}{\phi(v)}$ therefore:

$$z^{'} = \phi(-v)\phi(v)z \tag{54}$$

At this point Einstein observes that "since the relations between $x^{'}, y^{'}, z^{'}$ and x, y, z do not contain the time t, the systems K and $K^{'}$ are at rest with respect to one another, and it is clear that the transformation from K to $K^{'}$ must be the identical transformation". Recall that $K^{'}$ is moving at velocity $-v$ parallel to the moving system k, which is moving relative to the stationary system K with velocity v, and so is at rest with respect to K.

Thus:

$$\phi(v)\phi(-v) = 1 \tag{55}$$

To determine $\phi(v)$ Einstein considers that part of the axis of Y in system k which lies between $\xi = 0, \eta = 0, \zeta = 0$ and $\xi = 0, \eta = l, \zeta = 0$. In effect this part of the Y axis is a rod moving perpendicularly to its axis with velocity v relative to system K. In system K its ends have coordinates:

$$x_1 = vt \tag{56}$$

$$y_1 = \frac{l}{\phi(v)} \quad \text{see (26)} \tag{57}$$

$$z_1 = 0 \tag{58}$$

and

$$x_2 = vt \tag{59}$$

$$y_2 = 0 \tag{60}$$

$$z_2 = 0 \tag{61}$$

Hence the length of the rod measured in system K is $y_1 - y_2 = \frac{l}{\phi(v)}$ which gives meaning to $\phi(v)$. By symmetry considerations the length of a given rod moving perpendicularly to its axis as measured in the stationary system depends only on the velocity and not on the direction. This leads then to the conclusion that the length of the moving rod measured in the stationary system does not change if you interchange v and $-v$ ie

$$\frac{l}{\phi(v)} = \frac{l}{\phi(-v)} \tag{62}$$

which leads to:

$$\phi(v) = \phi(-v) \tag{63}$$

But from (55) we must then have:

$$\phi(v)^2 = 1 \tag{64}$$

Hence we take $\phi(v) = 1$ since lengths are positive.

So finally we get equations (24)-(27) in their final form:

$$\tau = \beta(t - \frac{vx}{c^2}) \tag{65}$$

$$\xi = \beta(x - vt) \tag{66}$$

$$\eta = y \tag{67}$$

$$\zeta = z \tag{68}$$

where $\beta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

5 Comments

As already noted, Einstein abandoned the method described above in the 1905 paper as a means for explaining special relativity and, after going through the process it is frankly easy to appreciate why. The development of the partial differential equation coupled with his transitive concept of simultaneity suggests to me what he was trying to do was to move from an infinitesimal local approach to simultaneity and then "stitch" together more widely spatially separated events.

It is therefore useful to compare the 1905 paper approach to how the transformations Einstein derived are commonly derived today. There is, in fact, a wide variety of approaches when you get into the detail. No doubt classroom experiences have suggested that certain approaches work better than others.

As already noted above, the concept that the speed of light in a vacuum is the same in all inertial frames does not seem consistent with experience since if light travels towards me at speed c and I run away from the same light at speed v, I ought to measure the speed of the light as $c-v$. The Galilean transfromation laws for two inertial reference frames S and $S^{'}$ which move relative to each other with velocity $\mathbf{v} = (v, 0, 0)$ are related in the usual Cartesian coordinates by:

$$\begin{aligned} x^{'} =& x - vt \\ y^{'} =& y \\ z^{'} =& z \\ t^{'} =& t \end{aligned} \tag{69}$$

So if a light ray travels in the x direction in system S with speed c then it traces out a trajectory given by $x = ct$. From (69) we have:

$$\begin{aligned} x^{'} =& x - vt \\ =& ct - vt \\ =& (c-v)t^{'} \end{aligned} \tag{70}$$

So Galileo gives us the relative velocity of $c - v$ and if he is wrong there must be something wrong with the transformations in (69). The idea is to find a transformation that is consistent with both postulates so something has to give and that is the concept of absolute time reflected by $t^{'} = t$. In what follows we stick with two inertial systems S and $S^{'}$ moving with relative speed v and we ignore the y, z directions which are perpendicular to the direction of motion. We denote the Cartesian coordinate frame for S by (x, t) and for $S^{'}$ by $(x^{'}, t^{'})$. The most general functional form would be something like:

$$\begin{aligned} x^{'} =& f(x,t) \\ t^{'} =& g(x,t) \end{aligned} \tag{71}$$

However, because of the assumption of inertial frames, a particle in such a frame will travel with a constant velocity so the trajectory in the (x, t) plane is a straight line. Since S and $S^{'}$ are inertial frames straight lines are mapped to straight lines and hence we are looking for a linear map so f, g must have the following forms:

$$\begin{aligned} x^{'} =& \alpha_1 x + \alpha_2 t \\ t^{'} =& \alpha_3 x + \alpha_4 t \end{aligned} \tag{72}$$

where α_i can each be a function of v for $i = 1, 2, 3, 4$.

The origin of $S^{'}$, $x^{'} = 0$, moves the trajectory $x = vt$ in frame S. If we assume that the origin $x^{'} = 0$ coincides with $x = 0$ when $t = 0$ then the straight lines $x = vt$ map to $x^{'} = 0$.

Thus from the first equation in (72) we have:

$$0 = \alpha_1 vt + \alpha_2 t \implies \alpha_1 v = -\alpha_2 \tag{73}$$

Hence:

$$x^{'} = \alpha_1(x - vt) = \gamma(v)(x - vt) \tag{74}$$

Now $\gamma(v) = \gamma(-v)$ ie it is rotationally invariant any only magnitude matters but this can also be seen by doing what Einstein effectively did. Imagine two new inertial frames which are identical to S and $S^{'}$ (call them $\tilde{S}$ and $\tilde{S}^{'}$ respectively) except that the x coordinate is measured in the opposite direction ie $\tilde{x} = -x$ and $\tilde{x}^{'} = -x^{'}$. So S moves with velocity $+v$ relative to $S^{'}$ but $\tilde{S}$ moves with velocity $-v$ with respect to $\tilde{S}^{'}$. This means that:

$$\tilde{x}^{'} = \gamma(-v)(\tilde{x} + v\tilde{t}) \tag{75}$$

Looking at (74) and (75) this means that we must have $\gamma(v) = \gamma(-v)$.

Similarly, if we look at things from $S^{'}$ then relative to it frame S moves backwards with velocity $-v$ and so we get:

$$x = \gamma(-v)(x^{'} + vt^{'}) \tag{76}$$

By parity of reasoning as above $\gamma(v) = \gamma(-v)$. Some texts simply argue that by rotational invariance $\gamma(v)$ can only depend on the magnitude of the relative velocity rather than its direction, hence $\gamma(v)$ must be an even function.

The assumption that light has the same speed in all inertial frames means that in S a light ray has trajectory:

$$x = ct \tag{77}$$

and in $S^{'}$ it has trajectory:

$$x^{'} = ct^{'} \tag{78}$$

Substituting (77) and (78) into (74) and (75) we have:

$$t^{'} = \frac{\gamma(v)}{c}(c - v)t \tag{79}$$

$$t = \frac{\gamma(-v)}{c}(c + v)t^{'} = \frac{\gamma(v)}{c}(c + v)t^{'} \tag{80}$$

For consistency between (79) and (80) we have:

$$t^{'} = \frac{\gamma(v)}{c}(c - v)\frac{\gamma(v)}{c}(c + v)t^{'} \tag{81}$$

so we must have:

$$1 = \frac{\gamma(v)^2}{c^2}(c^2 - v^2) \quad \Longrightarrow \gamma(v) = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} \tag{82}$$

From (74) and (76) with $\gamma(v) = \gamma(-v)$ we have:

$$x^{'} = \gamma(v)(x - vt) \tag{83}$$

$$x = \gamma(v)(x^{'} + vt^{'}) \tag{84}$$

Therefore:

$$\begin{aligned}
x =& \gamma(v)^2(x - vt) + \gamma(v)vt^{'} \\
(1 - \gamma(v)^2)x =& -\gamma(v)^2 vt + \gamma(v)vt^{'} \\
t^{'} =& \frac{x(1 - \gamma(v)^2)}{\gamma(v)v} + \gamma(v)t \\
=& \frac{x}{\gamma(v)v}\left(1 - \frac{c^2}{c^2 - v^2}\right) + \gamma(v)t \\
=& \frac{x}{\gamma(v)v}\left(-\frac{v^2}{c^2 - v^2}\right) + \gamma(v)t \\
=& \frac{-xv}{\gamma(v)}\frac{\gamma(v)^2}{c^2} + \gamma(v)t \\
=& \gamma(v)\left(t - \frac{vx}{c^2}\right)
\end{aligned} \tag{85}$$

Thus in the one spatial dimension case we get the Lorentz transformation equations:

$$\boxed{\begin{aligned}
x^{'} =& \gamma(v)\left(x - vt\right) \\
t^{'} =& \gamma(v)\left(t - \frac{vx}{c^2}\right)
\end{aligned}} \tag{86}$$

One sees claims that special relativity is "easy' (not always by reference to the difficulties of general relativity) and that high school students can "easily" understand it. The people who make these bold claims perhaps should read the 1905 paper and wonder for a moment if it was so damn easy why an astonishing thinker like Einstein chose to develop it in the way he did, and why he never used the 1905 explanation as the paradigmatic form of explanation.

6 References

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7 History

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08/10/2017 - corrected typo on page 4 ("and then being reflected back to A" rather than "and then being reflected back to C")