The Laplace transform of a Gaussian

Peter Haggstrom
www.gotohaggstrom.com
mathsatbondibeach@gmail.com

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1 The Fourier transform of a Gaussian

Students of Fourier theory are aware that the Fourier transform of a Gaussian is a Gaussian. This important result can be proved in a couple of ways. Two proofs are set out in [1]. As a preliminary to the related Laplace transform case a short proof of the Fourier transform case runs as follows. We want to show that if:

$$f(x) = e^{-\pi x^2} \quad \text{then } \hat{f}(\xi) = f(\xi) \tag{1}$$

In what follows we use this form of the Fourier transform of $f(x)$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx \tag{2}$$

With this we let:

$$F(\xi) = \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, e^{-2\pi i x \xi}\, dx \tag{3}$$

So

$$F(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1 \tag{4}$$

This is just a standard result in first year calculus. In what follows we need these two results [see, [2] page 136]. The symbol $\xrightarrow{\mathcal{F}}$ simply means that the Fourier transform is being taken.

$$f^{'}(x) \xrightarrow{\mathcal{F}} 2\pi i \xi\, \hat{f}(\xi) \tag{5}$$

$$-2\pi i x\, f^{'}(x) \xrightarrow{\mathcal{F}} \frac{d}{d\xi}\hat{f}(\xi) \tag{6}$$

Therefore:

$$
\begin{aligned}
F^{'}(\xi) =& \int_{-\infty}^{\infty} f(x)\,(-2\pi i x) e^{-2\pi i x}\,dx \\
=& i\int_{-\infty}^{\infty} f^{'}(x)\,e^{-2\pi i x}\,dx \quad \text{since } f^{'}(x) = -2\pi x f(x) \\
=& i(2\pi i\xi)\hat{f}(\xi) \\
=& -2\pi\xi F(\xi)
\end{aligned} \tag{7}
$$

Define $G(\xi) = F(\xi)e^{\pi\xi^2}$. Therefore:

$$
\begin{aligned}
G^{'}(\xi) =& F(\xi) \times 2\pi\xi e^{\pi\xi^2} + e^{\pi\xi^2} F^{'}(\xi) \\
=& e^{\pi\xi^2}\left(F^{'}(\xi) + 2\pi\xi F(\xi)\right) \\
=& 0 \quad \text{using (7)}
\end{aligned} \tag{8}
$$

This means that G is a constant. Since $F(0) = 1$ it follows that $G \equiv 1$.

Hence:

$$
\begin{aligned}
1 =& F(\xi)e^{\pi\xi^2} \\
\therefore F(\xi) =& e^{-\pi\xi^2}
\end{aligned} \tag{9}
$$

2 The Laplace transform of a Gaussian

With the Fourier transform behind us we can now do the much easier Laplace transform. Recall that for $s > 0$ the Laplace transform is defined as follows:

$$
\mathcal{L}[f(x)](s) = \int_{0}^{\infty} f(x)e^{-sx}\,dx \tag{10}
$$

With $f(x) = e^{-\pi x^2}$ we have:

$$
\begin{aligned}
\mathcal{L}[e^{-\pi x^2}](s) &= \int_0^\infty e^{-\pi x^2} e^{-sx}\,dx \\
&= \int_0^\infty e^{-\pi(x^2+sx)}\,dx \\
&= \int_0^\infty e^{-\pi[(x+\frac{s}{2})^2-\frac{s^2}{4}]}\,dx \\
&= e^{\frac{\pi s^2}{4}} \int_0^\infty e^{-\pi(x+\frac{s}{2})^2)}\,dx \\
&= e^{\frac{\pi s^2}{4}} \int_{\frac{s}{2}}^\infty e^{-\pi u^2}\,du \quad \text{with substitution } u = x + \frac{s}{2} \\
&= e^{\frac{\pi s^2}{4}} \Big[\int_0^\infty e^{-\pi u^2}\,du - \int_0^{\frac{s}{2}} e^{-\pi u^2}\,du\Big]
\end{aligned}
\tag{11}
$$

We know that $\int_0^\infty e^{-\pi u^2}\,du = \frac{1}{2}$ and the Error Function erf(x) is defined as follows:

$$
\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \tag{12}
$$

The Complementary Error Function (erfc(x)) is defined as:

$$
\text{erfc(x)} = 1 - \text{erf(x)} \tag{13}
$$

With the substitution $t = \sqrt{\pi}u$ we have that:

$$
\begin{aligned}
\int_0^{\frac{s}{2}} e^{-\pi u^2}\,du &= \frac{1}{\sqrt{\pi}} \int_0^{\frac{s\sqrt{\pi}}{2}} e^{-t^2}\,dt \\
&= \frac{1}{2}\operatorname{erf}\big(\frac{s\sqrt{\pi}}{2}\big)
\end{aligned}
\tag{14}
$$

So going back to (11) we have that the Laplace transform of a Gaussian is:

$$
\begin{aligned}
\mathcal{L}[e^{-\pi x^2}](s) &= e^{\frac{\pi s^2}{4}} \Big[\frac{1}{2} - \frac{1}{2}\operatorname{erf}\big(\frac{s\sqrt{\pi}}{2}\big)\Big] \\
&= \frac{1}{2} e^{\frac{\pi s^2}{4}} \operatorname{erfc}\big(\frac{s\sqrt{\pi}}{2}\big)
\end{aligned}
\tag{15}
$$

So the Laplace transform of a Gaussian is not a Gaussian.

3 References

[1] Peter Haggstrom, *"Basic Fourier integrals"*, `https://www.gotohaggstrom.com/Basic%20Fourier%20integrals.pdf`

[2] Elias M Stein and Rami Shakarchi, *"Fourier Analysis: An Introduction",* Princeton University Press, 2003

4 History

Created 28/11/2018

29/11/2018 - corrected a dopey typo in (2)!