# The Laplacian in curvilinear coordinates - the full story

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## 1 Introduction

In this article I provide some background to Laplace's equation (and hence the Laplacian ) as well as giving detailed derivations of the Laplacian in various coordinate systems using several different techniques.

So where does Laplace's equation come from? Following Newton's work on gravitation and many other things the methods of calculus were applied in the $18^{th}$ and $19^{th}$ centuries to gravitational, electrostatic, electromagnetic, hydrodynamic, acoustic and many other phenomena. Laplace's monumental treatise on celestial mechanics contains the equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$ in the context of the behaviour of fluids as the following extract from a translation of the treatise shows ( [6],page 237):

The equation $(K)$ relative to the continuity of the fluid, becomes

$$0 = \left\{\frac{d\rho}{dt}\right\} + \left\{\frac{d\rho}{dx}\right\}.\left\{\frac{d\varphi}{dx}\right\} + \left\{\frac{d\rho}{dy}\right\}.\left\{\frac{d\varphi}{dy}\right\} + \left\{\frac{d\rho}{dz}\right\}.\left\{\frac{d\varphi}{dz}\right\}$$

$$+\rho.\left\{\left\{\frac{d^2\varphi}{dx^2}\right\} + \left\{\frac{d^2\varphi}{dy^2}\right\} + \left\{\frac{d^2\varphi}{dz^2}\right\}\right\};$$

consequently, we shall have in the case of homogenous fluids,

$$0 = \left\{\frac{d^2\varphi}{dx^2}\right\} + \left\{\frac{d^2\varphi}{dy^2}\right\} + \left\{\frac{d^2\varphi}{dz^2}\right\}.$$

The full derivation of Laplace's equation in the context of fluids is quite complex the way he did it and can be found at [6, pages 222-237]. A shorter derivation in the context of heat flow can be found in [4, pages 76-78] and runs as follows. Kellogg's book was used for many years by students at MIT and other universities. I will reproduce the derivation but there is one step the author glosses over which is quite fundamental to an understanding of partial differential equations in a physical modelling context.

It is supposed we have a solid all of whose points are not at the same temperature. The rate of flow of heat is represented by a vector $(u, v, w)$ whose direction at any point is that in which the heat is flowing and whose magnitude is obtained by taking an element $\Delta S$ of the plane through the point $P$, say, normal to the direction of flow, then determining the number of calories (CGS system) or Joules (SI system ) per second flowing through the element and dividing this number by the area $\Delta S$ and then finally taking the limit of the resulting quotient as the maximum chord of $\Delta S$ approaches 0. In what follows I will stick with references to calories.

It is usually assumed that the velocity of the heat flow is proportional to the rate of fall of the temperature, $U$, at $P$. This constant of proportionality depends of the nature of the material and is referred to as conductivity. The next assumption is that the material is thermally isotropic which means that the conductivity has no directional bias. On this basis it is to be expected that the flow vector is in the same direction as the gradient of the temperature but with the opposite sign (heat flows from warmer places to cooler places). Thus we have:

$$\begin{aligned} u &= -k\frac{\partial U}{\partial x} \\ v &= -k\frac{\partial U}{\partial y} \\ w &= -k\frac{\partial U}{\partial z} \end{aligned} \tag{1}$$

where $k$ is the conductivity factor which is assumed constant for homogeneous materials. Note that in Laplace's derivation of his equation for fluids showcased above, the factor $\rho$ is assumed to be constant for a homogeneous fluid. The flow field is always normal to the isothermal surfaces $U$ = constant. This is because the gradient is normal to such surfaces.

The next physical assumption involves considering a region $T$ in the body and balancing the rate of the flow of the heat into it against the rise in temperature. The rate of flow of heat into $T$ in calories per second is the negative of the flux of the field $(u, v, w)$ out of the bounding surface i.e:

$$-\iint_S V_n\, dS = -\iint_S (ul + vm + wn)\, dS \tag{2}$$

$V_n$ is the component of the velocity of heat flow in the direction of the outward normal to the surface and $l, m, n$ are the respective direction cosines relating the the $x, y, z$ axes. If the specific heat of the material is $c$ then a unit of heat will raise the temperature of a unit mass by $c$ degrees. Hence the number of calories per second received per unit mass is:

$$c\frac{\partial U}{\partial t} \tag{3}$$

The number of calories per second received by the whole mass in $T$ is therefore:

$$\iiint_T c\rho \frac{\partial U}{\partial t}\, dV \tag{4}$$

where $\rho$ is the density per unit mass of the material.

The Divergence Theorem allows us to assert that:

$$\iiint_T \operatorname{div}\mathbf{V}\, dV = \iint_S V_n\, dS \tag{5}$$

Hence from (2):

$$\begin{aligned} \iint_S V_n\, dS &= -\iiint_T \operatorname{div}\mathbf{V}\, dV \\ &= -\iiint_T (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})\, dV \end{aligned} \tag{6}$$

But:

$$\iint_S V_n\, dS = \iiint_T c\rho \frac{\partial U}{\partial t}\, dV \tag{7}$$

Therefore using (6) and (7):

$$\iiint_T (c\rho \frac{\partial U}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})\, dV = 0 \tag{8}$$

Assuming the integrand of (8) is continuous, the integrand must vanish since the integral vanishes for every region $T$. That this is the case follows from the fundamental lemma of the calculus of variations which, at it its simplest, says that if $f(x)$ is continuous on $[a, b]$ and if $\int_a^b f(x)\, h(x)\, dx =$ for every continuous function $h(x)$ on $[a, b]$ such that $h(a) = h(b) = 0$, then $f(x) = 0$ for all $x \in [a, b]$. This logic extends to three dimensions as is the case in (8). A proof of the one dimensional case can be found in [3, page 9]. The gist of the proof that the integrand of (8) vanishes runs as follows. For the purposes of a contradiction we suppose our region is a small cube in which the integrand is non-zero, say, positive at some point. Because of continuity there is a neighbourhood of this point where the integrand is positive and so the integral over this small, arbitrary cube must be positive, thereby producing a contradiction. This is essentially what Hilbert and Courant do in their proof of the one dimensional case [2, page 185].

Thus we have come to this physical assumption:

$$\frac{\partial U}{\partial t} = -\frac{1}{c\rho}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) \tag{9}$$

Kellogg says the following [ 4,page 78]:

"The flow of heat in a body may be stationary ie such that the temperature at each point is independent of the time. Such, for instance, might be the situation in a bar, wrapped with insulating material, one end of which is kept in boiling water, and the other end in ice-water. Though the heat would be constantly flowing, the temperature might not vary sensibly with the time”

Let’s just take this for granted at the moment but it deserves much more explanation than this throwaway line.

Substituting (1) into (9) we have (assuming $c, k, \rho$ are constants):

$$\begin{aligned}\frac{\partial U}{\partial t} =& -\frac{1}{c\rho}\Big(\frac{\partial}{\partial x}\big(-k\frac{\partial U}{\partial x}\big) + \frac{\partial v}{\partial y}\big(-k\frac{\partial U}{\partial y}\big) + \frac{\partial}{\partial z}\big(-k\frac{\partial U}{\partial z}\big)\Big)\\ =& a^2\big(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\big)\end{aligned} \tag{10}$$

where $a^2 = \frac{k}{c\rho}$.

On the assumption of stationarity we have $\frac{\partial U}{\partial t} = 0$ and Laplace’s equation results:

$$\boxed{\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0} \tag{11}$$

Let us now return to Kellogg’s quoted statement above concerning stationarity ie lack of variation in time. On the surface of things this is a bold claim since one would expect the heat to move over time giving rise to a variation in temperature with respect to time.

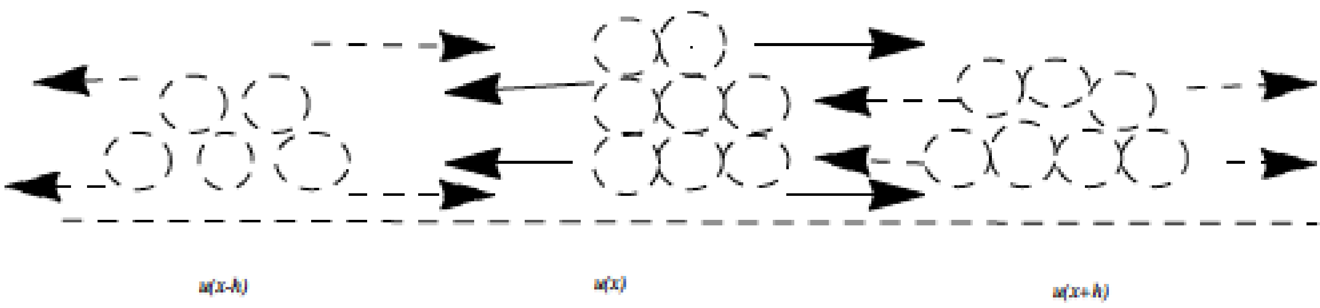
The heat equation is such an important equation in physics it is worth understanding some dimensions to it which are not to my knowledge covered generally in undergraduate engineering courses or indeed physics courses. What follows is an expansion of some comments made by the well known partial differential equation expert Luis Cafarelli of the University of Texas, Austin in his plenary lecture at the 2013 Mathematical Congress of the Americas (MCA) at Guanajuato, Mexico. The one dimensional heat equation ie $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ represents a diffusion process. A diffusion process such as that represented by the heat equation has a tendency to revert to its surrounding average. To see how this might be the case we need to look at the most simple situation – ie one dimension, which indicates a relationship between diffusion and the Laplacian (in $n$ dimensions the Laplacian is $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$).

In one dimension the Laplacian of $u$ is simply the second derivative of u and so we look at the limit of the second order incremental quotient. Recall that:

$$u''(x) = \lim_{h\to 0} \frac{\frac{u(x+h)-u(x)}{h} - \Big(\frac{u(x)-u(x-h)}{h}\Big)}{h} = \lim_{h\to 0} \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \tag{12}$$

In the diagram below the balls represent particles which can jump left and right in proportion to their number in a pile. The pile at $x$ gains half of the particles coming from adjacent piles and loses its own. This simple rule gives rise to a balance equation of gains (ie $\frac{1}{2}u(x-h) + \frac{1}{2}u(x+h)$) minus losses (ie $u(x)$ )which is proportional to:

$$\frac{1}{2}\Big(u(x+h) + u(x-h) - 2u(x)\Big) \tag{13}$$




Equation (13) looks suspiciously like (12) - hence the connection with the Laplacian which has remarkable features: it is rotationally invariant, independent of the system of coordinates and represents a diffusion. As we go up in dimensions we consider the Laplacian as a limit gain-loss of density $u$ at $x$. We take the average over a unit sphere S of the radial second derivatives in every direction and one of the fundamental results of harmonic analysis is that:

$$\Delta(u) = \fint_S u_{rr}\, dA(s) \tag{14}$$

Recall that for a function $u$ defined in a ball $B(x,r)$ of radius $r$ about $x$ in $\mathbb{R}^n$, with boundary $\partial B(x,r)$ and $\alpha(n)$ is the volume of a unit ball in $\mathbb{R}^n$ and $n\alpha(n)$ is the surface area of the unit ball in $\mathbb{R}^n$, the average of $u$ on $B(x,r)$ is:

$$\fint_{B(x,r)} u(y)\, dy = \frac{1}{\alpha(n)r^n} \int_{B(x,r)} u(y)\, dy \tag{15}$$

In 2 dimensions a function $u$ is harmonic at P (ie it satisfies Laplace's equation $\Delta = 0$) if and only if:

$$u(P) = \frac{1}{2\pi r} \int_{\partial B(P,r)} u\, ds = \frac{1}{\pi r^2} \int_{B(P,r)} u\, dx\, dy \tag{16}$$

To prove (16) we take $P = (x_0, y_0)$ and we suppose that $u(x_0,y_0) = \frac{1}{2\pi r} \int_{\partial B(P,r)} u(x,y)\, ds$ then:

$$\begin{aligned} u(x_0,y_0) = \frac{1}{2\pi r} \int_{\partial B(P,r)} u(x,y)\, ds &= \frac{1}{2\pi r} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\, rd\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\, d\theta \end{aligned} \tag{17}$$

The LHS of (17) is simply a constant so if we differentiate with respect to $r$ under the integral sign (and use the chain rule) we get:

$$\begin{aligned} 0 = \frac{1}{2\pi} \int_0^{2\pi} (\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta)\, d\theta &= \frac{1}{2\pi r} \int_0^{2\pi} (\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta)\, rd\theta \\ &= \frac{1}{2\pi r} \int_{\partial B(P,r)} \nabla u \,.\, \nu\, ds = \frac{1}{2\pi r} \int_{B(P,r)} div\,(\nabla u)\, dy\, dx = \frac{1}{2\pi r} \int_{B(P,r)} \Delta u\, dy\, dx \end{aligned} \tag{18}$$

The divergence theorem justifies the last step in (18). Hence based on our assumption $u(x_0, y_0) = \frac{1}{2\pi r}\int_{\partial B(P,r)} u(x,y)\,ds$ we have shown that $0 = \int_{B(P,r)} \Delta u\,dy\,dx$ for all $r > 0$. If all this holds for every $P$ in some open subset $\Omega$ in $\mathbb{R}^2$ then we must have that $\Delta u = 0$ for each such $P$. Thus $u$ is harmonic.

What this averaging suggests is that the heat equation $\frac{\partial u}{\partial t} = \Delta(u)$ reflects the fact that the density $u$ at the point $x$ makes an infinitesimal comparison within its neighbourhood and tries to revert to the surrounding average.

It is this local behaviour that justifies Kellogg's claim concerning stationarity which, at first blush, seems counterintuitive.

## 2 The taxonomy of PDEs

In the taxonomy of partial differential equations (PDEs) the broad breakdown is:

Laplace's equation: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Elliptic

Wave equation: $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ Hyperbolic

Heat equation: : $\frac{\partial u}{\partial t} - \gamma \frac{\partial^2 u}{\partial x^2} = 0$ Parabolic

Hyperbolic and parabolic PDEs typically model dynamic phenomena and hence have a time dimension. PDEs which model equilibrium phenomena (where time has in effect decayed away) are generally elliptic in nature and involve only spatial dimensions as in the case of the Laplace and Poisson equations. Elliptic PDEs are associated with boundary value problems whereas parabolic and hyperbolic PDEs are associated with initial value and initial boundary value problems ([7] section 4.4)

The classification of linear second order PDEs arises from the form of the "discriminant" of the general form of the PDE. Writing $u_{xx}$ for $\frac{\partial^2 u}{\partial x^2}$ etc the general form of a linear second order PDE in two variables $x$ and $y$ is:

$$L[u] = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{19}$$

A,B,. . . G can all be functions of $x$ and $y$ and if $G \equiv 0$ the PDE is homogeneous.

What drives the classification is the discriminant which is defined as:

$$\Delta = B^2 – 4AC \tag{20}$$

The discriminant is connected with the general quadratic equation where it is assumed that not all of the coefficients are zero:

$$Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{21}$$

The classification becomes:

Hyperbolic if $\Delta > 0$

Parabolic if $\Delta = 0$

Elliptic if $\Delta < 0$

Equation (21) can be written in matrix form as:

$$X^t M X + N X + F = 0 \tag{22}$$

where the matrix M has the form:

$$M = \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \tag{23}$$

By the Spectral Theorem (see [1] pages 256-257) there is an orthogonal matrix $P$ such that $PMP^t$ diagonalises M and one can then apply a translation to eliminate as much as possible the linear and constant terms $NX + F$. A change of variable $X' = PX$ or $X = P^t X'$ is then made. The equation is thus reduced to one of the canonical forms for an ellipse, hyperbola or parabola. Note that $-4 \det M = B^2 - 4AC$.

Rather than using $x, y$ etc we can formulate the general linear PDE as follows:

$$\underbrace{\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}}_{\text{Principal Part}} + \sum_{i,j=1}^{n} b_i u_{x_i} + cu = d \tag{24}$$

Assuming continuity of the second partials we will have $u_{x_i x_j} = u_{x_j x_i}$ and the $n \times n$ matrix $\mathbf{A} = (a_{ij})$ will be symmetric and so $a_{ij} = a_{ji}$. We can then diagonalise the principal part of (24).

As an example we can find an orthogonal transformation which eliminates the mixed partials in the following equation:

$$2u_{x_1x_1} + 2u_{x_2x_2} - 15u_{x_3x_3} + 8u_{x_1x_2} - 12u_{x_2x_3} - 12u_{x_1x_3} = 0 \tag{25}$$

The matrix corresponding to the principal part of (25) is:

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & -6 \\ 4 & 2 & -6 \\ -6 & -6 & -15 \end{pmatrix} \tag{26}$$

The eigenvalues of $\mathbf{A}$ are $-2, -18$ and 9 and the orthogonal matrix $\mathbf{B}$ we want is found to be:

$$\mathbf{B} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ \frac{-1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{2}{3} \\ 0 & \frac{4}{3\sqrt{2}} & \frac{-1}{3} \end{pmatrix} \tag{27}$$

Thus we will have:

$$\mathbf{C} = \mathbf{B^t A B} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & 9 \end{pmatrix} \tag{28}$$

If we make a change of variables $\xi = \mathbf{B^t x}$ (25) transforms to:

$$-2u_{\xi_1\xi_1} - 18u_{\xi_2\xi_2} + 9u_{\xi_3\xi_3} = 0 \tag{29}$$

We then let:

$$\begin{aligned} \eta_1 &= \frac{\xi_1}{\sqrt{2}} \\ \eta_2 &= \frac{\xi_2}{3\sqrt{2}} \\ \eta_3 &= \frac{\xi_3}{3} \end{aligned} \tag{30}$$

With these changes and multiplying (29) by $-1$ and noting that $u_{\xi_i\xi_i} = \frac{\partial}{\partial \xi_i}\left(\frac{\partial u}{\partial \xi_i}\right)$ etc we get:

$$u_{\eta_1\eta_1} + u_{\eta_2\eta_2} - u_{\eta_3\eta_3} = 0 \tag{31}$$

This is an hyperbolic PDE. More details are set out in the appendix.

# 3 The Laplacian in polar coordinates – brute force

There are various ways of deriving the Laplacian in curvilinear coordinates such a polar, cylindrical or spherical. This is a purely mathematical exercise unrelated to the underlying physics of, say, electostatics. The first method is simply one of brute force and it gets increasingly more complicated as we move from polar, to cylindrical and then spherical coordinates. There are many other curvilinear coordinate systems beyond those just mentioned eg parabolic cylindrical, paraboloidal, elliptic cylindrical etc. There is nothing deep about the process but it is error prone. It also suffers from a lack of intuition which makes it hard to remember the form of the Laplacian. Later on I show how to remember a general formula for the Laplacian.

A second method is favoured in a physics and engineering contexts and it involves divergences and directional derivatives. This approach is followed in [8] and many other textbooks.

The third method is completely general and has it roots in tensor calculus/differential geometry.

In this article I systematically go through each method.

In applying the chain rule it is crucial to remember that the partial derivatives are calculated holding other variables constant. Because the chain rule is generally written without explicit reference to the variables being held constant it is very easy to make errors that will build upon one another. Thus, for instance, the chain rule can be written explicitly this way:

$$\left(\frac{\partial u}{\partial x}\right)_{\text{y constant}} = \left(\frac{\partial u}{\partial r}\right)_{\theta\text{ constant}} \left(\frac{\partial r}{\partial x}\right)_{\text{y constant}} + \left(\frac{\partial u}{\partial \theta}\right)_{\text{r constant}} \left(\frac{\partial \theta}{\partial x}\right)_{\text{y constant}} \tag{32}$$

$\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$ are evaluated at constant y and $\frac{\partial r}{\partial y}$ and $\frac{\partial \theta}{\partial y}$ are evaluated at constant x.

The trap is to do this.

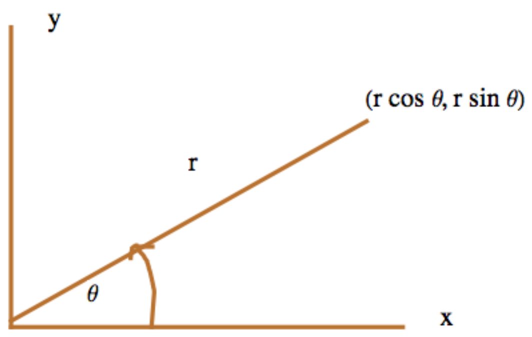
$$
\begin{aligned}
x &= r\cos\theta \\
\implies r &= \frac{x}{\cos\theta} \\
\implies \frac{\partial r}{\partial x} &= \frac{1}{\cos\theta}
\end{aligned}
\tag{33}
$$

This is actually $\left(\frac{\partial r}{\partial x}\right)_{\theta \text{ constant}}$. Keep these principles in mind in what follows.

The relevant equations are:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\theta &= \arctan\left(\frac{y}{x}\right) \\
r &= \sqrt{x^2 + y^2}
\end{aligned}
\tag{35}
$$




In preparation for applying the chain rule we now need to derive the following derivatives and we assume equality of mixed partials:

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2)^{\frac{-1}{2}} \times 2x = \frac{x}{r} = \cos\theta \tag{36}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \tag{37}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{-y}{x^2} = \frac{-y}{r^2} = \frac{-\sin\theta}{r} \tag{38}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{r^2} = \frac{\cos\theta}{r} \tag{39}$$

We now need to calculate $\frac{\partial u}{\partial x}$ and then $\frac{\partial^2 u}{\partial x^2}$ and similarly for $\frac{\partial u}{\partial y}$ and then $\frac{\partial^2 u}{\partial y^2}$.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \end{aligned} \tag{40}$$

Hence:

$$\begin{aligned} \frac{\partial u^2}{\partial x^2} =& \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}\Big) \\ =& \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial x}\Big)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial x}\Big)\frac{\partial \theta}{\partial x} \\ =& \cos\theta\frac{\partial}{\partial r}\Big[\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\Big] - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\Big[\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\Big] \\ =& \cos\theta\Big[\cos\theta\frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin\theta}{r^2}\frac{\partial u}{\partial\theta}\Big] - \frac{\sin\theta}{r}\Big[-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial r\partial\theta} \\ & - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2} - \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\Big] \\ =& \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial^2 u}{\partial\theta^2} \end{aligned} \tag{41}$$

We now replicate the process for $\frac{\partial^2 u}{\partial y^2}$.

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \end{aligned} \tag{42}$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} =& \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y}\Big) \\
&= \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial y}\Big)\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial y}\Big)\frac{\partial \theta}{\partial y} \\
&= \sin\theta \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial y}\Big) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial y}\Big) \\
&= \sin\theta \frac{\partial}{\partial r}\Big(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\Big) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\Big(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\Big) \\
&= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&- \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} \\
&= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}
\end{aligned} \tag{43}
$$

So adding (41) and (43) we have:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} =& \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&+ \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
=& \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}
\end{aligned} \tag{44}
$$

So the Laplacian operator in polar form can be written as follows:

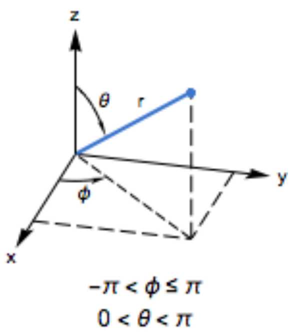
$$
\boxed{\Delta_{\text{polar}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}} \tag{45}
$$

## 4 The Laplacian in spherical coordinates - brute force

There is nothing in principle difficult about this but it is incredibly easy to make a slip. You will only do this once in your life and you will appreciate some of the other methods covered later in this article. We start with the basic coordinate relationships and list the relevant derivatives:

$$
\begin{aligned}
x &= r\,\sin\theta\,\cos\phi \\
y &= r\,\sin\theta\,\sin\phi \\
z &= r\,\cos\theta
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arccos(\frac{z}{r}) \\
\phi &= \arctan(\frac{y}{x})
\end{aligned} \tag{47}
$$




The Laplacian in rectangular coordinates is:

$$
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{48}
$$

You will also see $\nabla^2$ used as the symbol for the Laplacian.

We now get each component of (48):

$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} \tag{49}
$$

$$
\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} \tag{50}
$$

$$
\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z} \tag{51}
$$

We now need the various sub-components such as $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$ etc and then we will need to work out the second derivatives. The basic components are given below:

$$
\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r} = \sin\theta\,\cos\phi \tag{52}
$$

$$
\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\,\sin\phi \tag{53}
$$

$$
\frac{\partial r}{\partial z} = \cos\theta \tag{54}
$$

$$\begin{aligned}\frac{\partial\theta}{\partial x} &= \frac{\partial}{\partial x}\arccos(\frac{z}{r}) = \frac{\partial}{\partial x}\arccos(\frac{z}{\sqrt{x^2+y^2+z^2}}) \\ &= \frac{-1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}} \times \frac{-2zx}{2(x^2+y^2+z^2)^{\frac{3}{2}}} \\ &= \frac{zx}{r^2\sqrt{x^2+y^2}} \\ &= \frac{r^2\sin\theta\cos\theta\cos\phi}{r^2\sqrt{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}} \\ &= \frac{1}{r}\cos\theta\cos\phi\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{\partial\theta}{\partial y} &= \frac{\partial}{\partial y}\arccos(\frac{z}{r}) = \frac{\partial}{\partial y}\arccos(\frac{z}{\sqrt{x^2+y^2+z^2}}) \\ &= \frac{-1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}} \times \frac{-2zy}{2(x^2+y^2+z^2)^{\frac{3}{2}}} \\ &= \frac{zy}{r^2\sqrt{x^2+y^2}} \\ &= \frac{r^2\sin\theta\cos\theta\sin\phi}{r^2\sqrt{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}} \\ &= \frac{1}{r}\cos\theta\sin\phi\end{aligned} \tag{56}$$

$$\begin{aligned}\frac{\partial\theta}{\partial z} &= \frac{\partial}{\partial z}\arccos(\frac{z}{r}) = \frac{\partial}{\partial z}\arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ &= \frac{-1}{\sqrt{1-\frac{z^2}{x^2+y^2+z^2}}} \times \frac{x^2+y^2}{r^3} \\ &= \frac{-(x^2+y^2)}{r^2\sqrt{x^2+y^2}} \\ &= \frac{-1}{r}\sin\theta\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{\partial}{\partial x}\arctan(\frac{y}{x}) \\ &= \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2} \\ &= \frac{-y}{x^2+y^2} \\ &= \frac{-\sin\phi}{r\sin\theta}\end{aligned} \tag{58}$$

$$\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{\partial}{\partial y}\arctan(\frac{y}{x}) \\
&= \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} \\
&= \frac{x}{x^2+y^2} \\
&= \frac{\cos\phi}{r\sin\theta}
\end{aligned} \tag{59}$$

$$\frac{\partial \phi}{\partial z} = 0 \tag{60}$$

We start with $\frac{\partial u}{\partial x}$ in terms of the derivatives with respect to the spherical coordinates (using (52)-(60) ):

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} \\
&= \sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial u}{\partial \phi}
\end{aligned} \tag{61}$$

Now we work out $\frac{\partial^2 u}{\partial x^2}$:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial x}\right)\frac{\partial \phi}{\partial x} \\
&= \underbrace{\sin\theta\cos\phi\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)}_{A} + \underbrace{\frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)}_{B} - \underbrace{\frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial x}\right)}_{C}
\end{aligned} \tag{62}$$

$$\begin{aligned}
A &= \sin\theta\cos\phi\,\frac{\partial}{\partial r}\Big[\sin\theta\cos\phi\,\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\,\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial u}{\partial \phi}\Big] \\
&= \sin\theta\cos\phi\Big[\sin\theta\cos\phi\,\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cos\theta\cos\phi\,\frac{\partial^2 u}{\partial r\partial \theta} - \frac{1}{r^2}\cos\theta\cos\phi\,\frac{\partial u}{\partial \theta} + \frac{1}{r^2}\frac{\sin\phi}{\sin\theta}\frac{\partial u}{\partial \phi} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial^2 u}{\partial r\partial \phi}\Big] \\
&= \sin^2\theta\cos^2\phi\,\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\sin\theta\cos\theta\cos^2\phi\,\frac{\partial^2 u}{\partial r\partial \theta} - \frac{1}{r^2}\sin\theta\cos\theta\cos^2\phi\,\frac{\partial u}{\partial \theta} + \frac{1}{r^2}\sin\phi\cos\phi\,\frac{\partial u}{\partial \phi} \\
&\quad - \frac{1}{r}\sin\phi\cos\phi\,\frac{\partial^2 u}{\partial r\partial \phi}
\end{aligned} \tag{63}$$

Hence:

$$\boxed{\begin{aligned}
A = \sin^2\theta\cos^2\phi\,\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\sin\theta\cos\theta\cos^2\phi\,\frac{\partial^2 u}{\partial r\partial \theta} - \frac{1}{r^2}\sin\theta\cos\theta\cos^2\phi\,\frac{\partial u}{\partial \theta} + \frac{1}{r^2}\sin\phi\cos\phi\,\frac{\partial u}{\partial \phi} \\
- \frac{1}{r}\sin\phi\cos\phi\,\frac{\partial^2 u}{\partial r\partial \phi}
\end{aligned}}$$

$$
\begin{aligned}
B =& \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta}\Big(\frac{\partial u}{\partial x}\Big)\\
=& \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta}\Big[\sin\theta\cos\phi\,\frac{\partial u}{\partial r}+\frac{1}{r}\cos\theta\cos\phi\,\frac{\partial u}{\partial\theta}-\frac{1}{r}\frac{\sin\phi}{\sin\theta}\,\frac{\partial u}{\partial\phi}\Big]\\
=& \frac{1}{r}\cos\theta\cos\phi\Big[\cos\theta\cos\phi\frac{\partial u}{\partial r}+\sin\theta\cos\phi\,\frac{\partial^2 u}{\partial\theta\partial r}-\frac{1}{r}\sin\theta\cos\phi\,\frac{\partial u}{\partial\theta}+\frac{1}{r}\cos\theta\cos\phi\,\frac{\partial^2 u}{\partial\theta^2}\\
&-\frac{1}{r}\frac{\sin\phi}{\sin\theta}\,\frac{\partial^2 u}{\partial\theta\partial\phi}+\frac{1}{r}\frac{\sin\phi\cos\theta}{\sin^2\theta}\,\frac{\partial u}{\partial\phi}\Big]\\
=& \frac{1}{r}\cos^2\theta\cos^2\phi\,\frac{\partial u}{\partial r}+\frac{1}{r}\sin\theta\cos\theta\cos^2\phi\,\frac{\partial^2 u}{\partial\theta\partial r}-\frac{1}{r^2}\sin\theta\cos\theta\cos^2\phi\,\frac{\partial u}{\partial\theta}+\frac{1}{r^2}\cos^2\theta\cos^2\phi\,\frac{\partial^2 u}{\partial\theta^2}\\
&-\frac{1}{r^2}\frac{\cos\theta\sin\phi\cos\phi}{\sin\theta}\,\frac{\partial^2 u}{\partial\theta\partial\phi}+\frac{1}{r^2}\frac{\cos^2\theta\sin\phi\cos\phi}{\sin^2\theta}\,\frac{\partial u}{\partial\phi}
\end{aligned}
\tag{64}
$$

Hence:

$$
\boxed{
\begin{aligned}
B =& \frac{1}{r}\cos^2\theta\cos^2\phi\,\frac{\partial u}{\partial r}+\frac{1}{r}\sin\theta\cos\theta\cos^2\phi\,\frac{\partial^2 u}{\partial\theta\partial r}-\frac{1}{r^2}\sin\theta\cos\theta\cos^2\phi\,\frac{\partial u}{\partial\theta}+\frac{1}{r^2}\cos^2\theta\cos^2\phi\,\frac{\partial^2 u}{\partial\theta^2}\\
&-\frac{1}{r^2}\frac{\cos\theta\sin\phi\cos\phi}{\sin\theta}\,\frac{\partial^2 u}{\partial\theta\partial\phi}+\frac{1}{r^2}\frac{\cos^2\theta\sin\phi\cos\phi}{\sin^2\theta}\,\frac{\partial u}{\partial\phi}
\end{aligned}
}
$$

$$
\begin{aligned}
C =& -\frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\Big(\frac{\partial u}{\partial x}\Big)\\
=& -\frac{1}{r}\frac{\sin\phi}{\sin\theta}\,\frac{\partial}{\partial\phi}\Big[\sin\theta\cos\phi\frac{\partial u}{\partial r}+\frac{1}{r}\cos\theta\cos\phi\frac{\partial u}{\partial\theta}-\frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial u}{\partial\phi}\Big]\\
=& -\frac{1}{r}\frac{\sin\phi}{\sin\theta}\Big[-\sin\theta\sin\phi\,\frac{\partial u}{\partial r}+\sin\theta\cos\phi\,\frac{\partial^2 u}{\partial\phi\partial r}-\frac{1}{r}\cos\theta\sin\phi\,\frac{\partial u}{\partial\theta}+\frac{1}{r}\cos\theta\cos\phi\,\frac{\partial^2 u}{\partial\phi\partial\theta}\\
&-\frac{1}{r}\frac{\cos\phi}{\sin\theta}\,\frac{\partial u}{\partial\phi}-\frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial^2 u}{\partial\phi^2}\Big]\\
=& \frac{1}{r}\sin^2\phi\frac{\partial u}{\partial r}-\frac{1}{r}\sin\phi\cos\phi\frac{\partial^2 u}{\partial\phi\partial r}+\frac{\sin^2\phi\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial\theta}-\frac{\sin\phi\cos\phi\cos\theta}{r^2\sin\theta}\,\frac{\partial^2 u}{\partial\phi\partial\theta}\\
&+\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}+\frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}
\end{aligned}
\tag{65}
$$

Hence:

$$
\boxed{
\begin{aligned}
C =& \frac{1}{r}\sin^2\phi\frac{\partial u}{\partial r}-\frac{1}{r}\sin\phi\cos\phi\frac{\partial^2 u}{\partial\phi\partial r}+\frac{\sin^2\phi\cos\theta}{r^2\sin\theta}\frac{\partial u}{\partial\theta}-\frac{\sin\phi\cos\phi\cos\theta}{r^2\sin\theta}\frac{\partial^2 u}{\partial\phi\partial\theta}\\
&+\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}+\frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}
\end{aligned}
}
$$

We now repeat the process to arrive at $\frac{\partial^2 u}{\partial y^2}$.

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} \\
&= \sin\theta\sin\phi\,\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\,\frac{\partial u}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial u}{\partial \phi}
\end{aligned}
\tag{66}
$$

As before we have:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial y}\Big)\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial y}\Big)\frac{\partial \theta}{\partial y} + \frac{\partial}{\partial \phi}\Big(\frac{\partial u}{\partial y}\Big)\frac{\partial \phi}{\partial y} \\
&= \underbrace{\sin\theta\sin\phi\frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial y}\Big)}_{D} + \underbrace{\frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial y}\Big)}_{E} + \underbrace{\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\Big(\frac{\partial u}{\partial y}\Big)}_{F}
\end{aligned}
\tag{67}
$$

Now we work out each component:

$$
\begin{aligned}
D &= \sin\theta\sin\phi\,\frac{\partial}{\partial r}\Big[\sin\theta\sin\phi\,\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\,\frac{\partial u}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial u}{\partial \phi}\Big] \\
&= \sin\theta\sin\phi\Big[\sin\theta\sin\phi\,\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\cos\theta\sin\phi\,\frac{\partial u}{\partial \theta} + \frac{1}{r}\cos\theta\sin\phi\,\frac{\partial^2 u}{\partial r\partial \theta} - \frac{\cos\phi}{r^2\sin\theta}\,\frac{\partial u}{\partial \phi} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial^2 u}{\partial r\partial \phi}\Big] \\
&= \sin^2\theta\sin^2\phi\,\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\sin\theta\cos\theta\sin^2\phi\,\frac{\partial u}{\partial \theta} + \frac{1}{r}\sin\theta\cos\theta\sin^2\phi\,\frac{\partial^2 u}{\partial r\partial \theta} - \frac{1}{r^2}\sin\phi\cos\phi\,\frac{\partial u}{\partial \phi} \\
&\quad + \frac{1}{r}\sin\phi\cos\phi\,\frac{\partial^2 u}{\partial r\partial \phi}
\end{aligned}
\tag{68}
$$

So:

$$
\boxed{
\begin{aligned}
D = \sin^2\theta\sin^2\phi\,\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\sin\theta\cos\theta\sin^2\phi\,\frac{\partial u}{\partial \theta} + \frac{1}{r}\sin\theta\cos\theta\sin^2\phi\,\frac{\partial^2 u}{\partial r\partial \theta} - \frac{1}{r^2}\sin\phi\cos\phi\,\frac{\partial u}{\partial \phi} \\
+ \frac{1}{r}\sin\phi\cos\phi\,\frac{\partial^2 u}{\partial r\partial \phi}
\end{aligned}
}
$$

$$
\begin{aligned}
E =& \frac{1}{r}\cos\theta\sin\phi\,\frac{\partial}{\partial\theta}\Big(\frac{\partial u}{\partial y}\Big) \\
=& \frac{1}{r}\cos\theta\sin\phi\,\frac{\partial}{\partial\theta}\Big[\sin\theta\sin\phi\,\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\,\frac{\partial u}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial u}{\partial\phi}\Big] \\
=& \frac{1}{r}\cos\theta\sin\phi\Big[\cos\theta\sin\phi\,\frac{\partial u}{\partial r} + \sin\theta\sin\phi\,\frac{\partial^2 u}{\partial\theta\partial r} - \frac{1}{r}\sin\theta\sin\phi\,\frac{\partial u}{\partial\theta} + \frac{1}{r}\cos\theta\sin\phi\,\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial^2 u}{\partial\theta\partial\phi} \\
&- \frac{\cos\phi\cos\theta}{r\sin^2\theta}\,\frac{\partial u}{\partial\phi}\Big] \\
=& \frac{1}{r}\cos^2\theta\sin^2\phi\,\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\sin\theta\sin^2\phi\,\frac{\partial^2 u}{\partial\theta\partial r} - \frac{1}{r^2}\cos\theta\sin\theta\sin^2\phi\,\frac{\partial u}{\partial\theta} + \frac{1}{r^2}\cos^2\theta\sin^2\phi\,\frac{\partial^2 u}{\partial\theta^2} \\
&+ \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\,\frac{\partial^2 u}{\partial\theta\partial\phi} - \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\,\frac{\partial u}{\partial\phi}
\end{aligned}
\tag{69}
$$

Hence:

$$
\boxed{
\begin{aligned}
E = \frac{1}{r}\cos^2\theta\sin^2\phi\,\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\sin\theta\sin^2\phi\,\frac{\partial^2 u}{\partial\theta\partial r} - \frac{1}{r^2}\cos\theta\sin\theta\sin^2\phi\,\frac{\partial u}{\partial\theta} + \frac{1}{r^2}\cos^2\theta\sin^2\phi\,\frac{\partial^2 u}{\partial\theta^2} \\
+ \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\,\frac{\partial^2 u}{\partial\theta\partial\phi} - \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\,\frac{\partial u}{\partial\phi}
\end{aligned}
}
$$

$$
\begin{aligned}
F =& \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial\phi}\Big(\frac{\partial u}{\partial y}\Big) \\
=& \frac{\cos\phi}{r\sin\theta}\,\frac{\partial}{\partial\phi}\Big[\sin\theta\sin\phi\,\frac{\partial u}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\,\frac{\partial u}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial u}{\partial\phi}\Big] \\
=& \frac{\cos\phi}{r\sin\theta}\Big[\sin\theta\cos\phi\,\frac{\partial u}{\partial r} + \sin\theta\sin\phi\,\frac{\partial^2 u}{\partial\phi\partial r} + \frac{1}{r}\cos\theta\cos\phi\,\frac{\partial u}{\partial\theta} + \frac{1}{r}\cos\theta\sin\phi\,\frac{\partial^2 u}{\partial\phi\partial\theta} \\
&- \frac{\sin\phi}{r\sin\theta}\,\frac{\partial u}{\partial\phi} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial^2 u}{\partial\phi^2}\Big] \\
=& \frac{1}{r}\cos^2\phi\,\frac{\partial u}{\partial r} + \frac{1}{r}\sin\phi\cos\phi\,\frac{\partial^2 u}{\partial\phi\partial r} + \frac{\cos\theta\cos^2\phi}{r^2\sin\theta}\,\frac{\partial u}{\partial\theta} + \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\,\frac{\partial^2 u}{\partial\phi\partial\theta} \\
&- \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\,\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\,\frac{\partial^2 u}{\partial\phi^2}
\end{aligned}
\tag{70}
$$

So:

$$
\boxed{
\begin{aligned}
F = \frac{1}{r}\cos^2\phi\,\frac{\partial u}{\partial r} + \frac{1}{r}\sin\phi\cos\phi\,\frac{\partial^2 u}{\partial\phi\partial r} + \frac{\cos\theta\cos^2\phi}{r^2\sin\theta}\,\frac{\partial u}{\partial\theta} + \frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\,\frac{\partial^2 u}{\partial\phi\partial\theta} \\
- \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\,\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\,\frac{\partial^2 u}{\partial\phi^2}
\end{aligned}
}
$$

Finally we calculate $\frac{\partial^2 u}{\partial z^2}$:

$$
\begin{aligned}
\frac{\partial u}{\partial z} =& \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\underbrace{\frac{\partial \phi}{\partial z}}_{=0} \\
=& \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} \\
=& \cos\theta\,\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial u}{\partial \theta}
\end{aligned}
\tag{71}
$$

Hence:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial z^2} =& \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial z}\Big)\frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial z}\Big)\frac{\partial \theta}{\partial z} \\
=& \underbrace{\cos\theta\,\frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial z}\Big)}_{G}\underbrace{-\frac{1}{r}\sin\theta\,\frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial z}\Big)}_{H}
\end{aligned}
\tag{72}
$$

$$
\begin{aligned}
G =& \cos\theta\,\frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial z}\Big) \\
=& \cos\theta\,\frac{\partial}{\partial r}\Big[\cos\theta\,\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial u}{\partial \theta}\Big] \\
=& \cos\theta\Big[\cos\theta\,\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\sin\theta\,\frac{\partial u}{\partial \theta} - \frac{1}{r}\sin\theta\,\frac{\partial^2 u}{\partial r\partial \theta}\Big] \\
=& \cos^2\theta\,\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\sin\theta\cos\theta\,\frac{\partial u}{\partial \theta} - \frac{1}{r}\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial r\partial \theta}
\end{aligned}
\tag{73}
$$

So:

$$
\boxed{G = \cos^2\theta\,\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\sin\theta\cos\theta\,\frac{\partial u}{\partial \theta} - \frac{1}{r}\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial r\partial \theta}}
$$

$$
\begin{aligned}
H =& -\frac{1}{r}\sin\theta\,\frac{\partial}{\partial \theta}\Big(\frac{\partial u}{\partial z}\Big) \\
=& -\frac{1}{r}\sin\theta\,\frac{\partial}{\partial \theta}\Big[\cos\theta\,\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial u}{\partial \theta}\Big] \\
=& -\frac{1}{r}\sin\theta\Big[-\sin\theta\,\frac{\partial u}{\partial r} + \cos\theta\,\frac{\partial^2 u}{\partial \theta\partial r} - \frac{1}{r}\cos\theta\,\frac{\partial u}{\partial \theta} - \frac{1}{r}\sin\theta\,\frac{\partial^2 u}{\partial \theta^2}\Big] \\
=& \frac{1}{r}\sin^2\theta\,\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial \theta\partial r} + \frac{1}{r^2}\sin\theta\cos\theta\,\frac{\partial u}{\partial \theta} + \frac{1}{r^2}\sin^2\theta\,\frac{\partial^2 u}{\partial \theta^2}
\end{aligned}
\tag{74}
$$

$$
\boxed{H = \frac{1}{r}\sin^2\theta\,\frac{\partial u}{\partial r} - \frac{1}{r}\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial \theta\partial r} + \frac{1}{r^2}\sin\theta\cos\theta\,\frac{\partial u}{\partial \theta} + \frac{1}{r^2}\sin^2\theta\,\frac{\partial^2 u}{\partial \theta^2}}
$$

Thus $\Delta_{\text{spherical}} = A + B + C + D + E + F + G + H$ and to see the final result of this we collect coefficients of the various derivatives (and assume equality of mixed partials):

$$
\begin{aligned}
\text{Coefficient of } \frac{\partial u}{\partial \theta} = & \underbrace{-\frac{2}{r^2}\sin\theta\cos\theta\cos^2\phi}_{\text{A and B}} + \underbrace{\frac{\sin^2\phi\cos\theta}{r^2\sin\theta}}_{C} \underbrace{-\frac{1}{r^2}\sin\theta\cos\theta\sin^2\phi}_{D} \\
& \underbrace{-\frac{1}{r^2}\cos\theta\sin\theta\sin^2\phi}_{E} + \underbrace{\frac{\cos\theta\cos^2\phi}{r^2\sin\theta}}_{F} + \underbrace{\frac{2}{r^2}\sin\theta\cos\theta}_{\text{G and H}} \\
= & \frac{-2\sin^2\theta\cos\theta\cos^2\phi + \sin^2\phi\cos\theta - \sin^2\theta\cos\theta\sin^2\phi - \cos\theta\sin^2\theta\sin^2\phi}{r^2\sin\theta} \\
& + \frac{\cos\theta\cos^2\phi + 2\sin^2\theta\cos\theta}{r^2\sin\theta} \\
= & \frac{-2\sin^2\theta\cos\theta\cos^2\phi + \cos\theta - \sin^2\theta\cos\theta\sin^2\phi - \cos\theta\sin^2\theta\sin^2\phi}{r^2\sin\theta} \\
& + \frac{2\sin^2\theta\cos\theta}{r^2\sin\theta} \\
= & \frac{2\sin^2\theta\cos\theta\sin^2\phi + \cos\theta - \sin^2\theta\cos\theta\sin^2\phi - \cos\theta\sin^2\theta\sin^2\phi}{r^2\sin\theta} \\
= & \frac{\sin^2\theta\cos\theta\sin^2\phi + \cos\theta - \sin^2\theta\cos\theta\sin^2\phi}{r^2\sin\theta} \\
= & \boxed{\frac{\cot\theta}{r^2}}
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
\text{Coefficient of } \frac{\partial u}{\partial \phi} = & \underbrace{\frac{1}{r^2}\sin\phi\cos\phi}_{A} + \underbrace{\frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}}_{B} + \underbrace{\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}}_{C} \underbrace{-\frac{1}{r^2}\sin\phi\cos\phi}_{D} \\
& \underbrace{-\frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}}_{E} \underbrace{-\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}}_{F} \\
= & \boxed{0}
\end{aligned} \tag{76}
$$

$$
\begin{aligned}
\text{Coefficient of } \frac{\partial^2 u}{\partial r\partial \theta} = & \underbrace{\frac{\sin\theta\cos\theta\cos^2\phi}{r}}_{A} + \underbrace{\frac{\sin\theta\cos\theta\cos^2\phi}{r}}_{B} + \underbrace{\frac{\sin\theta\cos\theta\sin^2\phi}{r}}_{D} \\
& \underbrace{+\frac{\cos\theta\sin\theta\sin^2\phi}{r}}_{E} \underbrace{-\frac{2\sin\theta\cos\theta}{r}}_{\text{G and H}} \\
= & \frac{2\sin\theta\cos\theta\cos^2\phi + 2\sin\theta\cos\theta\sin^2\phi - 2\sin\theta\cos\theta}{r} \\
= & \boxed{0}
\end{aligned} \tag{77}
$$

$$
\text{Coefficient of } \frac{\partial^2 u}{\partial r\partial \phi} = \frac{\overbrace{\sin\phi\cos\phi}^{D} \overbrace{- \sin\phi\cos\phi}^{A} \overbrace{+ \sin\phi\cos\phi}^{F} \overbrace{- \sin\phi\cos\phi}^{C}}{r} = \boxed{0} \tag{78}
$$

$$\begin{aligned}\text{Coefficient of } \frac{\partial^2 u}{\partial r^2} &= \underbrace{\sin^2\theta\cos^2\phi}_{A} + \underbrace{\sin^2\theta\sin^2\phi}_{D} + \underbrace{\cos^2\theta}_{G} \\ &= \boxed{1}\end{aligned} \tag{79}$$

$$\begin{aligned}\text{Coefficient of } \frac{\partial^2 u}{\partial \theta^2} &= \underbrace{\frac{1}{r^2}\cos^2\theta\cos^2\phi}_{B} + \underbrace{\frac{1}{r^2}\cos^2\theta\sin^2\phi}_{E} + \underbrace{\frac{1}{r^2}\sin^2\theta}_{H} \\ &= \boxed{\frac{1}{r^2}}\end{aligned} \tag{80}$$

$$\begin{aligned}\text{Coefficient of } \frac{\partial^2 u}{\partial \phi^2} &= \underbrace{\frac{\sin^2\phi}{r^2\sin^2\theta}}_{C} + \underbrace{\frac{\cos^2\phi}{r^2\sin^2\theta}}_{F} \\ &= \boxed{\frac{1}{r^2\sin^2\theta}}\end{aligned} \tag{81}$$

$$\begin{aligned}\text{Coefficient of } \frac{\partial u}{\partial r} &= \underbrace{\frac{\cos^2\theta\cos^2\phi}{r}}_{B} + \underbrace{\frac{\sin^2\phi}{r}}_{C} + \underbrace{\frac{\cos^2\theta\sin^2\phi}{r}}_{E} \\ &= + \underbrace{\frac{\cos^2\phi}{r}}_{F} + \underbrace{\frac{\sin^2\theta}{r}}_{H} \\ &= \frac{1}{r} + \frac{\cos^2\theta}{r}(\cos^2\phi + \sin^2\phi) + \frac{\sin^2\theta}{r} \\ &= \boxed{\frac{2}{r}}\end{aligned} \tag{82}$$

$$\begin{aligned}\text{Coefficient of } \frac{\partial^2 u}{\partial\theta\partial\phi} &= \underbrace{\frac{-\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}}_{B} \underbrace{\frac{-\sin\phi\cos\phi\cos\theta}{r^2\sin\theta}}_{C} + \underbrace{\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}}_{E} \\ &= + \underbrace{\frac{\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}}_{F} \\ &= \boxed{0}\end{aligned} \tag{83}$$

Putting this all together we finally get what we were after:

$$\boxed{\Delta_{\text{spherical}}\, u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}} \tag{84}$$

There are equivalent forms of (84) which are as follows:

$$\boxed{\Delta_{\text{spherical}}\, u = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2\,\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial u}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 u}{\partial\phi^2}} \tag{85}$$

This is verified as follows:

$$\begin{aligned}
\Delta_{\text{spherical}}\, u =& \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2\,\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial u}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 u}{\partial\phi^2} \\
=& \frac{1}{r^2}\Big[r^2\frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r}\Big] + \frac{1}{r^2\,\sin\theta}\Big[\sin\theta\frac{\partial^2 u}{\partial\theta^2} + \cos\theta\,\frac{\partial u}{\partial\theta}\Big] + \frac{1}{r^2\,\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \\
=& \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\,\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 u}{\partial\phi^2} + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\,\frac{\partial u}{\partial\theta}
\end{aligned} \tag{86}$$

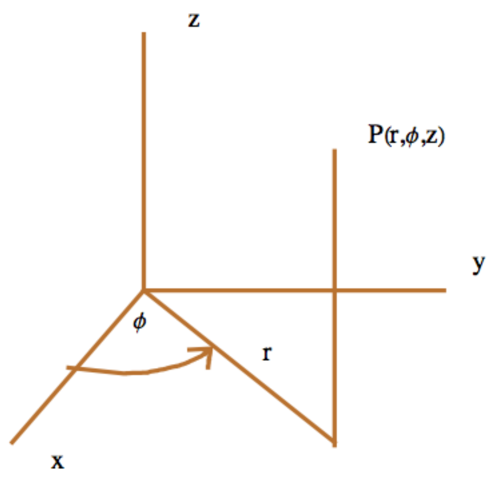
Another equivalent form is:

$$\boxed{\Delta_{\text{spherical}}\, u = \frac{1}{r}\,\frac{\partial^2}{\partial r^2}\big(ru\big) + \frac{1}{r^2\,\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial u}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 u}{\partial\phi^2}} \tag{87}$$

Note that some authors interchange $\theta$ and $\phi$ so that needs to be kept in mind (see the figure at the start of this section). For instance, if this occurs the form will be as follows (based on (87) ):

$$\boxed{\Delta_{\text{spherical}}\, u = \frac{1}{r}\,\frac{\partial^2}{\partial r^2}\big(ru\big) + \frac{1}{r^2\,\sin\phi}\,\frac{\partial}{\partial\phi}\Big(\sin\phi\,\frac{\partial u}{\partial\phi}\Big) + \frac{1}{r^2\sin^2\phi}\,\frac{\partial^2 u}{\partial\theta^2}} \tag{88}$$

# 5 The Laplacian in cylindrical coordinates – brute force




We start with the basic coordinate relationships and list the relevant derivatives:

$$\begin{aligned} x &= r\,\cos\phi \\ y &= r\,\sin\phi \\ z &= z \end{aligned} \tag{89}$$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \phi &= \arctan\left(\frac{y}{x}\right) \\ z &= z \end{aligned} \tag{90}$$

As before we work out the relevant building block derivatives:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{2x}{2\sqrt{x^2 + y^2}} \\ &= \frac{x}{\sqrt{x^2 + y^2}} \\ &= \frac{x}{r} \end{aligned} \tag{91}$$

$$
\begin{aligned}
\frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} \\
&= \frac{y}{r}
\end{aligned} \tag{92}
$$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{1}{\left(1+(\frac{y}{x})^2\right)} \times \frac{-y}{x^2} \\
&= -\frac{y}{x^2+y^2} \\
&= -\frac{y}{r^2}
\end{aligned} \tag{93}
$$

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{1}{\left(1+(\frac{y}{x})^2\right)} \times \frac{1}{x} \\
&= \frac{x}{x^2+y^2} \\
&= \frac{x}{r^2}
\end{aligned} \tag{94}
$$

Hence:

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial u}{\partial z}\underbrace{\frac{\partial z}{\partial x}}_{=0} \\
&= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} \\
&= \frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \phi}
\end{aligned} \tag{95}
$$

We now work out $\frac{\partial^2 u}{\partial x^2}$:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}\Big) \\
&= \frac{\partial}{\partial x}\Big(\frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \phi}\Big) \\
&= \underbrace{\frac{x}{r}\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial r}\Big)}_{A} + \underbrace{\frac{\partial u}{\partial r}\frac{\partial}{\partial x}\Big(\frac{x}{r}\Big)}_{B} \underbrace{- \frac{y}{r^2}\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial \phi}\Big)}_{C} \underbrace{- \frac{\partial u}{\partial \phi}\frac{\partial}{\partial x}\Big(\frac{y}{r^2}\Big)}_{D}
\end{aligned} \tag{96}
$$

$$
\begin{aligned}
A &= \frac{x}{r}\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial r}\Big) \\
&= \frac{x}{r}\Big[\frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial r}\Big)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\Big(\frac{\partial u}{\partial r}\Big)\frac{\partial \phi}{\partial x}\Big] \\
&= \frac{x}{r}\Big[\frac{x}{r}\frac{\partial^2 u}{\partial r^2} - \frac{y}{r^2}\frac{\partial^2 u}{\partial \phi\,\partial r}\Big] \\
&= \boxed{\frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{xy}{r^3}\frac{\partial^2 u}{\partial \phi\,\partial r}}
\end{aligned}
\tag{97}
$$

$$
\begin{aligned}
B &= \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\Big(\frac{x}{r}\Big) \\
&= \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\Big(\frac{x}{\sqrt{x^2+y^2}}\Big) \\
&= \frac{\partial u}{\partial r}\Big(\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\Big) \\
&= \boxed{\frac{y^2}{r^3}\frac{\partial u}{\partial r}}
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
C &= \frac{-y}{r^2}\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial \phi}\Big) \\
&= \frac{-y}{r^2}\Big[\frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial \phi}\Big)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\Big(\frac{\partial u}{\partial \phi}\Big)\frac{\partial \phi}{\partial x}\Big] \\
&= \frac{-y}{r^2}\Big[\frac{x}{r}\frac{\partial^2 u}{\partial r\,\partial \phi} - \frac{y}{r^2}\frac{\partial^2 u}{\partial \phi^2}\Big] \\
&= \boxed{\frac{-xy}{r^3}\frac{\partial^2 u}{\partial r\,\partial \phi} + \frac{y^2}{r^4}\frac{\partial^2 u}{\partial \phi^2}}
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
D &= -\frac{\partial u}{\partial \phi}\frac{\partial}{\partial x}\Big(\frac{y}{r^2}\Big) \\
&= -\frac{\partial u}{\partial \phi}\Big[\frac{\partial}{\partial r}\Big(\frac{y}{r^2}\Big)\frac{\partial r}{\partial x} + \underbrace{\frac{\partial}{\partial \phi}\Big(\frac{y}{r^2}\Big)}_{=0}\frac{\partial \phi}{\partial x}\Big] \\
&= -\frac{\partial u}{\partial \phi}\Big[\frac{x}{r}\frac{-2y}{r^3}\Big] \\
&= \boxed{\frac{2xy}{r^4}\frac{\partial u}{\partial \phi}}
\end{aligned}
\tag{100}
$$

Therefore:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} =& A + B + C + D \\
=& \frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{xy}{r^3}\frac{\partial^2 u}{\partial \phi\, \partial r} + \frac{y^2}{r^3}\frac{\partial u}{\partial r} - \frac{xy}{r^3}\frac{\partial^2 u}{\partial r \partial \phi} + \frac{y^2}{r^4}\frac{\partial^2 u}{\partial \phi^2} \\
&+ \frac{2xy}{r^4}\frac{\partial u}{\partial \phi} \\
=& \boxed{\frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{2xy}{r^3}\frac{\partial^2 u}{\partial \phi\, \partial r} + \frac{y^2}{r^3}\frac{\partial u}{\partial r} + \frac{y^2}{r^4}\frac{\partial^2 u}{\partial \phi^2} + \frac{2xy}{r^4}\frac{\partial u}{\partial \phi}}
\end{aligned}
\tag{101}
$$

We now repeat the process for $\frac{\partial^2 u}{\partial y^2}$:

$$
\begin{aligned}
\frac{\partial u}{\partial y} =& \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial u}{\partial z}\underbrace{\frac{\partial z}{\partial y}}_{=0} \\
=& \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} \\
=& \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial u}{\partial \phi}
\end{aligned}
\tag{102}
$$

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} =& \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y}\Big) \\
=& \frac{\partial}{\partial y}\Big(\frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial u}{\partial \phi}\Big) \\
=& \underbrace{\frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial r}\Big)}_{E} + \underbrace{\frac{\partial u}{\partial r}\frac{\partial}{\partial y}\Big(\frac{y}{\sqrt{x^2+y^2}}\Big)}_{F} \\
&+ \underbrace{\frac{x}{x^2+y^2}\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial \phi}\Big)}_{G} + \underbrace{\frac{\partial u}{\partial \phi}\frac{\partial}{\partial y}\Big(\frac{x}{x^2+y^2}\Big)}_{H}
\end{aligned}
\tag{103}
$$

$$
\begin{aligned}
E =& \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial r}\Big) \\
=& \frac{y}{\sqrt{x^2+y^2}}\Big[\frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial r}\Big)\frac{\partial r}{\partial y} + \frac{\partial}{\partial \phi}\Big(\frac{\partial u}{\partial r}\Big)\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\Big(\frac{\partial u}{\partial r}\Big)\underbrace{\frac{\partial z}{\partial y}}_{=0}\Big] \\
=& \frac{y}{\sqrt{x^2+y^2}}\Big(\frac{y}{\sqrt{x^2+y^2}}\frac{\partial^2 u}{\partial r^2} + \frac{x}{x^2+y^2}\frac{\partial^2 u}{\partial \phi\, \partial r}\Big) \\
=& \boxed{\frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{xy}{r^3}\frac{\partial^2 u}{\partial \phi\, \partial r}}
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
F =& \frac{\partial u}{\partial r}\frac{\partial}{\partial y}\Big(\frac{y}{\sqrt{x^2+y^2}}\Big) \\
=& \frac{\partial u}{\partial r}\Big(\frac{x^2+y^2-y^2}{r^3}\Big) \\
=& \boxed{\frac{x^2}{r^3}\frac{\partial u}{\partial r}}
\end{aligned} \tag{105}
$$

$$
\begin{aligned}
G =& \frac{x}{x^2+y^2}\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial \phi}\Big) \\
=& \frac{x}{x^2+y^2}\Big[\frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial \phi}\Big)\frac{\partial r}{\partial y} + \frac{\partial}{\partial \phi}\Big(\frac{\partial u}{\partial \phi}\Big)\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\Big(\frac{\partial u}{\partial \phi}\Big)\underbrace{\frac{\partial z}{\partial y}}_{=0}\Big] \\
=& \frac{x}{x^2+y^2}\Big[\frac{y}{\sqrt{x^2+y^2}}\frac{\partial^2 u}{\partial r\,\partial \phi} + \frac{x}{x^2+y^2}\frac{\partial^2 u}{\partial \phi^2}\Big] \\
=& \boxed{\frac{xy}{r^3}\frac{\partial^2 u}{\partial r\,\partial \phi} + \frac{x^2}{r^4}\frac{\partial^2 u}{\partial \phi^2}}
\end{aligned} \tag{106}
$$

$$
\begin{aligned}
H =& \frac{\partial u}{\partial \phi}\frac{\partial}{\partial y}\Big(\frac{x}{x^2+y^2}\Big) \\
=& \frac{\partial u}{\partial \phi}\frac{-2xy}{(x^2+y^2)^2} \\
=& \boxed{\frac{-2xy}{r^4}\frac{\partial u}{\partial \phi}}
\end{aligned} \tag{107}
$$

Therefore:

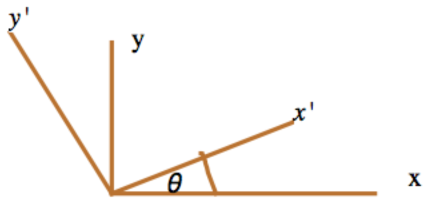
$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} =& E + F + G + H \\
=& \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{xy}{r^3}\frac{\partial^2 u}{\partial \phi\,\partial r} + \frac{x^2}{r^3}\frac{\partial u}{\partial r} + \frac{xy}{r^3}\frac{\partial^2 u}{\partial r\,\partial \phi} + \frac{x^2}{r^4}\frac{\partial^2 u}{\partial \phi^2} \\
& - \frac{2xy}{r^4}\frac{\partial u}{\partial \phi} \\
=& \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 u}{\partial \phi\,\partial r} + \frac{x^2}{r^3}\frac{\partial u}{\partial r} + \frac{x^2}{r^4}\frac{\partial^2 u}{\partial \phi^2} - \frac{2xy}{r^4}\frac{\partial u}{\partial \phi}
\end{aligned} \tag{108}
$$

Finally, we have:

$$
\begin{aligned}
\Delta_{\text{cylindrical}} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\
&= \left(\frac{x^2+y^2}{r^2}\right)\frac{\partial^2 u}{\partial r^2} + \left(\frac{2xy}{r^3} - \frac{2xy}{r^3}\right)\frac{\partial^2 u}{\partial r\,\partial \phi} + \left(\frac{y^2}{r^4} + \frac{x^2}{r^4}\right)\frac{\partial^2 u}{\partial \phi^2} \\
&\quad + \left(\frac{y^2}{r^3} + \frac{x^2}{r^3}\right)\frac{\partial u}{\partial r} + \left(\frac{2xy}{r^4} - \frac{2xy}{r^4}\right)\frac{\partial u}{\partial \phi} \\
&= \boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}}
\end{aligned} \tag{109}
$$

## 5.1 Rotational invariance of the Laplacian in two dimensions

Rotational invariance is a fundamental property of the Laplacian and in 2 dimensions one can demonstrate it as follows.




The transformation equations are as follows:

$$
\begin{aligned}
x' &= x\cos\theta + y\sin\theta \\
y' &= -\,x\sin\theta + y\cos\theta
\end{aligned} \tag{110}
$$

In matrix form this is:

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{111}
$$

Having got this far there are no horrors of principle in what follows. What we want to show is this:

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} \tag{112}
$$

As before we start off with $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in terms of $x'$ and $y'$:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} \\ &= \cos\theta\,\frac{\partial u}{\partial x'} - \sin\theta\,\frac{\partial u}{\partial y'} \\ &= U_x\end{aligned} \tag{113}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} \\ &= \sin\theta\,\frac{\partial u}{\partial x'} + \cos\theta\,\frac{\partial u}{\partial y'} \\ &= U_y\end{aligned} \tag{114}$$

Hence we have:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial U_x}{\partial x} \\ &= \frac{\partial U_x}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial U_x}{\partial y'}\frac{\partial y'}{\partial x} \\ &= \cos\theta\,\frac{\partial U_x}{\partial x'} - \sin\theta\,\frac{\partial U_x}{\partial y'} \\ &= \cos\theta\,\frac{\partial}{\partial x'}\Big(\cos\theta\,\frac{\partial u}{\partial x'} - \sin\theta\,\frac{\partial u}{\partial y'}\Big) - \sin\theta\,\frac{\partial}{\partial y'}\Big(\cos\theta\,\frac{\partial u}{\partial x'} - \sin\theta\,\frac{\partial u}{\partial y'}\Big) \\ &= \cos^2\theta\,\frac{\partial^2 u}{\partial x'^2} + \sin^2\theta\,\frac{\partial^2 u}{\partial y'^2} - \sin\theta\,\cos\theta\,\frac{\partial^2 u}{\partial x'\,\partial y'} - \sin\theta\,\cos\theta\,\frac{\partial^2 u}{\partial y'\,\partial x'}\end{aligned} \tag{115}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial U_y}{\partial y} \\ &= \frac{\partial U_y}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial U_y}{\partial y'}\frac{\partial y'}{\partial y} \\ &= \sin\theta\,\frac{\partial U_y}{\partial x'} + \cos\theta\,\frac{\partial U_y}{\partial y'} \\ &= \sin\theta\,\frac{\partial}{\partial x'}\Big(\sin\theta\,\frac{\partial u}{\partial x'} + \cos\theta\,\frac{\partial u}{\partial y'}\Big) + \cos\theta\,\frac{\partial}{\partial y'}\Big(\sin\theta\,\frac{\partial u}{\partial x'} + \cos\theta\,\frac{\partial u}{\partial y'}\Big) \\ &= \sin^2\theta\,\frac{\partial^2 u}{\partial x'^2} + \cos^2\theta\,\frac{\partial^2 u}{\partial y'^2} + \sin\theta\,\cos\theta\,\frac{\partial^2 u}{\partial x'\,\partial y'} + \sin\theta\,\cos\theta\,\frac{\partial^2 u}{\partial y'\,\partial x'}\end{aligned} \tag{116}$$

Thus:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (\sin^2\theta + \cos^2\theta)\Big(\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2}\Big) = \frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} \tag{117}$$

What we have just shown is that the Laplacian is invariant under a 2-dimensional orthogonal transformation $\mathbf{x}^{'} = P(\theta)\,\mathbf{x}$ where $P(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Note that if we had combined a translation with the rotation so that the transformation equations were of the following form, where $a, b$ are constants, the invariance still holds because the derivatives of the constants are zero:

$$\begin{aligned} x^{'} &= x\cos\theta + y\sin\theta + a \\ y^{'} &= -x\sin\theta + y\cos\theta + b \end{aligned} \tag{118}$$

Now to show that the Laplacian is rotationally invariant under a 3-dimensional orthogonal rotation one can somewhat laboriously replicate the 2-dimensional logic. Thus one could start with $\mathbf{y} = P\,\mathbf{x}$ where P is a $3 \times 3$ orthogonal matrix where $P = [p_{ij}]$. Then:

$$y_i = \sum_{j=1}^{n} p_{ij}\, x_j \tag{119}$$

$$\frac{\partial y_i}{\partial x_j} = p_{ij} \tag{120}$$

Then:

$$\begin{aligned} \frac{\partial\, u(P\mathbf{x})}{\partial x_1} &= \sum_{k=1}^{3} \frac{\partial\, u(\mathbf{y})}{\partial y_k}\,\frac{\partial y_k}{\partial y_1} \\ &= \sum_{k=1}^{3} p_{k\,1}\,\frac{\partial\, u(\mathbf{y})}{\partial y_k} \end{aligned} \tag{121}$$

After some labour we find that:

$$\begin{aligned} \frac{\partial^2\, u(P\mathbf{x})}{\partial {x_1}^2} =& p_{11}^2\,\frac{\partial^2 u(\mathbf{y})}{\partial {y_1}^2} + p_{11}p_{21}\,\frac{\partial^2 u(\mathbf{y})}{\partial y_1\,\partial y_2} + p_{11}p_{13}\,\frac{\partial^2 u(\mathbf{y})}{\partial y_1\,\partial y_3} \\ &+ p_{21}p_{11}\,\frac{\partial^2 u(\mathbf{y})}{\partial y_2\,\partial y_1} + p_{21}^2\,\frac{\partial^2 u(\mathbf{y})}{\partial {y_2}^2} + p_{21}p_{31}\,\frac{\partial^2 u(\mathbf{y})}{\partial y_2\,\partial y_3} \\ &+ p_{31}p_{11}\,\frac{\partial^2 u(\mathbf{y})}{\partial y_3\,\partial y_1} + p_{31}p_{21}\,\frac{\partial^2 u(\mathbf{y})}{\partial y_2\,\partial y_3} + p_{31}^2\,\frac{\partial^2 u(\mathbf{y})}{\partial {y_3}^2} \end{aligned} \tag{122}$$

If we keep going we ultimately get this:

$$\underbrace{(p_{11}^2 + p_{12}^2 + p_{13}^2)}_{=1}\,\frac{\partial^2 u(\mathbf{y})}{\partial {y_1}^2} + \underbrace{(p_{21}^2 + p_{22}^2 + p_{23}^2)}_{=1}\,\frac{\partial^2 u(\mathbf{y})}{\partial {y_2}^2} + \underbrace{(p_{31}^2 + p_{32}^2 + p_{33}^2)}_{=1}\,\frac{\partial^2 u(\mathbf{y})}{\partial {y_3}^2} \tag{123}$$

The orthogonality of P ensures that the coefficients in (123) are 1. We can however get this result more compactly as follows by using matrix notation throughout. First of all, because $P = [p_{ij}]$ is an orthogonal matrix we have:

$$PP^T = I \tag{124}$$

The $(i, j)$ element of $PP^T$ is $\sum_{k=1}^{3} p_{ik}\, p_{jk}$ and because I is the identity matrix we must have:

$$\sum_{k=1}^{3} p_{ik}\, p_{jk} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{125}$$

Using matrix notation we can show the rotational invariance of the Laplacian in a few steps as follows. As before P is an $n \times n$ orthogonal matrix such that $PP^T = I$. For $\mathbf{x} \in \mathbf{R^3}$ we have:

$$v(\mathbf{x}) = u(P\mathbf{x}) \tag{126}$$

What we want to show is that $\Delta u = 0$ implies that $\Delta v = 0$. The $(i, j)$ element of P is $p_{ij}$. The partial derivative operator we write as $D_i = \frac{\partial}{\partial x_i}$ or $D_i = \frac{\partial}{\partial y_i}$ as the context requires. Note that if $v(\mathbf{x}) = u(P\mathbf{x})$ then the $j^{th}$ element of $P\mathbf{x}$ is $y_j = \sum_{k=1}^{3} p_{jk}\, x_k$. So $\frac{\partial y_j}{\partial x_i} = p_{ji}$

Thus we have:

$$D_i\, v(\mathbf{x}) = \sum_{k=1}^{3} D_k u(P\mathbf{x}) p_{ki} \tag{127}$$

Therefore:

$$D_{ij}\, v(\mathbf{x}) = \sum_{l=1}^{3} \sum_{k=1}^{3} D_{kl} u(P\mathbf{x}) p_{ki}\, p_{lj} \tag{128}$$

Hence:

$$\begin{aligned}
\Delta v(\mathbf{x}) &= \sum_{i=1}^{3} D_{ii} \\
&= \sum_{i=1}^{3} \sum_{l=1}^{3} \sum_{k=1}^{3} D_{kl} u(P\mathbf{x}) p_{ki}\, p_{li} \\
&= \sum_{l=1}^{3} \sum_{k=1}^{3} D_{kl} u(P\mathbf{x}) \Big( \sum_{i=1}^{3} p_{ki}\, p_{li} \Big) \\
&= \sum_{l=1}^{3} \sum_{k=1}^{3} D_{kl} u(P\mathbf{x})\, \delta_{kl} \\
&= \Delta u(P\mathbf{x}) \\
&= 0
\end{aligned} \tag{129}$$

Note that the above line of argument generalises to $d$ dimensions. One of the reasons that the Laplacian is so ubiquitous in physics is that it can be used to express physical laws which do not depend on a special or preferred position. The Laplace operator also has spherical symmetry so that solutions which only depend on radial displacement are invariant under rotations about a chosen point $\xi$. Let $u = \psi(r)$ where:

$$r = |\mathbf{x} - \xi| = \sqrt{\sum_{i=1}^{3}(x_i - \xi_i)^2} \tag{130}$$

We want to work out $\Delta u$ for this type of solution.

$$\begin{aligned}\frac{\partial u}{\partial x_i} &= \frac{d\psi}{dr}\frac{\partial r}{\partial x_i}\\ &= \psi^{'}(r)\,\frac{x_i - \xi_i}{r}\end{aligned} \tag{131}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x_i^2} &= \frac{\partial}{\partial x_i}\Big(\psi^{'}(r)\,\frac{x_i - \xi_i}{r}\Big)\\ &= \frac{x_i - \xi_i}{r}\frac{\partial}{\partial r}\Big(\psi^{'}(r)\,\frac{x_i - \xi_i}{r}\Big) + \frac{\psi^{'}(r)}{r}\\ &= \frac{(x_i - \xi_i)^2}{r}\Big[\frac{r\psi^{''}(r) - \psi^{'}(r)}{r^2}\Big] + \frac{\psi^{'}(r)}{r}\end{aligned} \tag{132}$$

Hence:

$$\begin{aligned}\Delta u &= \sum_{i=1}^{3}\left[\frac{(x_i - \xi_i)^2}{r}\Big(\frac{r\psi^{''}(r) - \psi^{'}(r)}{r^2}\Big) + \frac{\psi^{'}(r)}{r}\right]\\ &= \frac{r^2}{r}\Big(\frac{r\psi^{''}(r) - \psi^{'}(r)}{r^2}\Big) + 3\frac{\psi^{'}(r)}{r}\\ &= \psi^{''}(r) + 2\frac{\psi^{'}(r)}{r}\\ &= 0\end{aligned} \tag{133}$$

In relation to (132) note that if $w = g(r, x_i)$ then:

$$\frac{\partial w}{\partial x_i} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x_i} + \frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial x_i} = \frac{\partial g}{\partial r}\frac{\partial r}{\partial x_i} + \frac{\partial g}{\partial x_i} \tag{134}$$

For $n$ dimensions the formula follows immediately from (133) to be as follows:

$$\Delta u = \psi^{''}(r) + \frac{n-1}{r}\,\psi^{'}(r) = 0 \tag{135}$$

To solve $\psi^{''}(r) + \frac{n-1}{r}\psi^{'} = 0$ we seek a solution of the form $\psi^{'}(r) = Cr^k$ for $C$ a non-zero constant. Then:

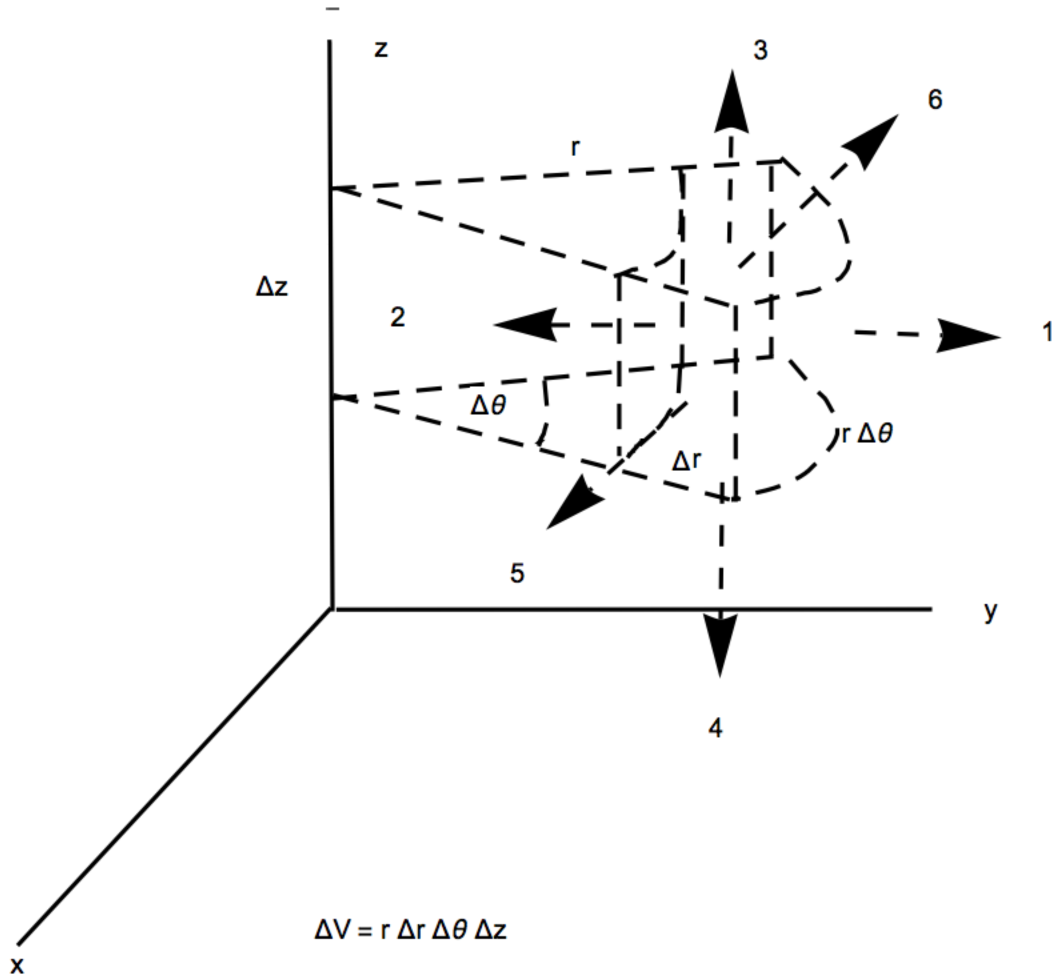
$$\begin{aligned} \psi^{''}(r) &= Ckr^{k-1} \\ \therefore Ckr^{k-1} + \frac{n-1}{r}Cr^k &= 0 \end{aligned} \tag{136}$$

which implies that $k = 1 - n$.

Integrating $\psi^{'}(r) = Cr^{1-n}$ we get $\psi(r) = \frac{C}{2-n}r^{2-n}$ if $n > 2$ or $\psi(r) = C\ln r$. For instance when $n = 3$ we get $\psi(r) = \frac{C}{r}$.

For more detail see [5], Chapter 4.

## 6 Derivation of the Laplacian in cylindrical coordinates using gradient and divergence techniques




The high level recipe for this line of approach comprises three steps (see [8]) :

1. Obtain an expression for the divergence of **F** ie:

$$\text{div}\,\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \tag{137}$$

where $F_r, F_\theta, F_z$ are the components of **F** in the drections of unit vectors $\hat{\mathbf{e}}_\mathbf{r}, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\mathbf{z}$ respectively.

To do this we start with this definition of the divergence:

$$\text{div}\,\mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint \mathbf{F}\,.\,\hat{\mathbf{n}}\,dS \tag{138}$$

where the volume limit is taken about a point $(x, y, z)$ and $\hat{\mathbf{n}}$ is a unit normal to the surface. Thus we need to look at an elementary volume in cylindrical coordinates and then calculate the relevant surface integral for each face (see the diagram above which has the unit normals indicated).

2. Obtain an expression for the gradient in cylindrical coordinates. To do this we start with a Taylor expansion of a scalar function $f(r, \theta, z)$. Thus the change in $f$ as we move from $(r, \theta, z)$ to $(r + \Delta r, \theta + \Delta\theta, z + \Delta z)$ is:

$$\Delta f = \frac{\partial f}{\partial r}\Delta r + \frac{\partial \theta}{\partial \theta}\Delta\theta + \frac{\partial f}{\partial z}\Delta z + \text{higher order terms} \tag{139}$$

Then we have to get an expression for $\Delta f$ in terms of the vectorial displacement $\Delta\mathbf{s}$. Recall that in Cartesian coordinates we will have:

$$\Delta\mathbf{s} = \Delta x\,\hat{\mathbf{e}}_{\mathbf{r}} + \Delta y\,\hat{\mathbf{e}}_{\theta} + \Delta z\,\hat{\mathbf{e}}_{\mathbf{z}} \tag{140}$$

where $\hat{\mathbf{e}}_{\mathbf{r}}, \hat{\mathbf{e}}_{\theta}, \hat{\mathbf{e}}_{\mathbf{z}}$ are the relevant unit basis vectors.

so that:

$$\begin{aligned} \Delta f =& \frac{\partial f}{\partial r}\Delta r + \frac{\partial \theta}{\partial \theta}\Delta\theta + \frac{\partial f}{\partial z}\Delta z + \text{higher order terms} \\ =& \Big(\frac{\partial f}{\partial x}\hat{\mathbf{e}}_{\mathbf{r}} + \frac{\partial f}{\partial y}\hat{\mathbf{e}}_{\theta} + \frac{\partial f}{\partial z}\hat{\mathbf{e}}_{\mathbf{z}}\Big)\,.\,\Delta\mathbf{s} \end{aligned} \tag{141}$$

But $\Delta\mathbf{s} = \hat{\mathbf{u}}\,\Delta s$ where $\hat{\mathbf{u}}$ is a unit vector in the direction of $\Delta\mathbf{s}$ and $\Delta s = |\Delta\mathbf{s}|$.

Therefore:

$$\begin{aligned} \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} =& \frac{df}{ds} = \Big(\frac{\partial f}{\partial x}\hat{\mathbf{e}}_{\mathbf{r}} + \frac{\partial f}{\partial y}\hat{\mathbf{e}}_{\theta} + \frac{\partial f}{\partial z}\hat{\mathbf{e}}_{\mathbf{z}}\Big)\,.\,\hat{\mathbf{u}} \\ =& (\nabla f)\,.\,\hat{\mathbf{u}} \end{aligned} \tag{142}$$

3. Finally, the Laplacian is:

$$\Delta f = \nabla^2 f = \nabla\,.\,(\nabla f) \tag{143}$$

We start with an elementary volume $\Delta V$ in cylindrical coordinates. This is a curved wedge of height $\Delta z$, radial length $\Delta r$ and angular arc length $(r + \Delta r)\Delta\theta = r\Delta\theta + \Delta r\Delta\theta = r\Delta\theta$ to first order (see the above diagram). Hence:

$$\Delta V = r\Delta r\,\Delta\theta\,\Delta z \tag{144}$$

If $S_i$ represents a surface (see above diagram ) with outward normal $\hat{\mathbf{n}}$ then we have the following, assuming a wedge centred at $(r, \theta, z)$.

$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iint_{S_1} F_r\, dS \\ &\approx F_r\left(r + \frac{\Delta r}{2}, \theta, z\right)\left(r + \frac{\Delta r}{2}\right)\Delta\theta\Delta z \end{aligned} \tag{145}$$

$$\begin{aligned} \iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= -\iint_{S_2} F_r\, dS \\ &\approx -F_r\left(r - \frac{\Delta r}{2}, \theta, z\right)\left(r - \frac{\Delta r}{2}\right)\Delta\theta\Delta z \end{aligned} \tag{146}$$

Therefore:

$$\begin{aligned} \frac{1}{\Delta V}\iint_{S_1+S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \frac{1}{r\Delta r\Delta\theta\Delta z}\left[F_r\left(r + \frac{\Delta r}{2}, \theta, z\right)\left(r + \frac{\Delta r}{2}\right) - F_r\left(r - \frac{\Delta r}{2}, \theta, z\right)\left(r - \frac{\Delta r}{2}\right)\right]\Delta\theta\Delta z \\ &= \frac{1}{r\Delta r}\left[F_r\left(r + \frac{\Delta r}{2}, \theta, z\right)\left(r + \frac{\Delta r}{2}\right) - F_r\left(r - \frac{\Delta r}{2}, \theta, z\right)\left(r - \frac{\Delta r}{2}\right)\right] \end{aligned} \tag{147}$$

Therefore as $\Delta V \to 0$, $\Delta r \to 0$:

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V}\iint_{S_1+S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \frac{1}{r}\frac{\partial}{\partial r}\left(rF_r\right) \tag{148}$$

We now replicate the same process for surfaces $S_3$ and $S_4$.

$$\begin{aligned} \iint_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iint_{S_3} F_z\, dS \\ &\approx F_z\left(r, \theta, z + \frac{\Delta z}{2}\right)r\Delta r\Delta\theta \end{aligned} \tag{149}$$

$$\begin{aligned} \iint_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= -\iint_{S_4} F_z\, dS \\ &\approx -F_z\left(r, \theta, z - \frac{\Delta z}{2}\right)r\Delta r\Delta\theta \end{aligned} \tag{150}$$

Therefore:

$$\begin{aligned} \frac{1}{\Delta V}\iint_{S_3+S_4} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \frac{1}{r\Delta r\Delta\theta\Delta z}\left[F_z\left(r, \theta, z + \frac{\Delta z}{2}\right) - F_z\left(r, \theta, z - \frac{\Delta z}{2}\right)\right]r\Delta r\Delta\theta \\ &= \frac{1}{\Delta z}\left[F_z\left(r, \theta, z + \frac{\Delta z}{2}\right) - F_z\left(r, \theta, z - \frac{\Delta z}{2}\right)\right] \end{aligned} \tag{151}$$

Hence:

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{S_3+S_4} \mathbf{F} \,.\, \hat{\mathbf{n}}\, dS = \frac{\partial F_z}{\partial z} \tag{152}$$

Finally we do the calculation for $S_5$ and $S_6$.

$$\begin{aligned} \iint_{S_6} \mathbf{F} \,.\, \hat{\mathbf{n}}\, dS &= \iint_{S_6} F_\theta\, dS \\ &\approx F_\theta\Big(r, \theta + \frac{\Delta\theta}{2}, z\Big)\Delta r \Delta z \end{aligned} \tag{153}$$

$$\begin{aligned} \iint_{S_5} \mathbf{F} \,.\, \hat{\mathbf{n}}\, dS &= - \iint_{S_5} F_\theta\, dS \\ &\approx - F_\theta\Big(r, \theta - \frac{\Delta\theta}{2}, z\Big)\Delta r \Delta z \end{aligned} \tag{154}$$

Therefore:

$$\begin{aligned} \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{S_5+S_6} \mathbf{F} \,.\, \hat{\mathbf{n}}\, dS &=\approx \frac{1}{r\Delta r \Delta\theta \Delta z}\Big[F_\theta\Big(r, \theta + \frac{\Delta\theta}{2}, z\Big) - F_\theta\Big(r, \theta - \frac{\Delta\theta}{2}, z\Big)\Big]\,\Delta r \Delta z \\ &= \frac{1}{r\Delta\theta}\Big[F_\theta\Big(r, \theta + \frac{\Delta\theta}{2}, z\Big) - F_\theta\Big(r, \theta - \frac{\Delta\theta}{2}, z\Big)\Big] \end{aligned} \tag{155}$$

Therefore:

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{S_5+S_6} \mathbf{F} \,.\, \hat{\mathbf{n}}\, dS = \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} \tag{156}$$

So putting it all together we have:

$$\boxed{\operatorname{div} \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}\Big(rF_r\Big) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}} \tag{157}$$

In accordance with the recipe we now need to find the gradient in cylindrical coordinates. If $f(r, \theta, z)$ we apply Taylor's theorem to obtain the change in $f$ due to a displacement from $(r, \theta, z)$ to $(r + \Delta r, \theta + \Delta\theta, z + \Delta z)$:

$$\Delta f = \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial \theta}\Delta\theta + \frac{\partial f}{\partial z}\Delta z + \text{higher order terms} \tag{158}$$

To write $\Delta f$ in terms of the vectorial displacement $\Delta\mathbf{s}$ we need to express $\Delta\mathbf{s}$ in terms of its natural basis as follows:

$$\Delta \mathbf{s} = \hat{\mathbf{e}}_{\mathbf{r}} \Delta r + \hat{\mathbf{e}}_{\theta} r \Delta\theta + \hat{\mathbf{e}}_{\mathbf{z}} \Delta z \tag{159}$$

In relatiion to (159), recall that the displacement in the direction of increasing $\theta$ is $(r+\Delta r)\Delta\theta = r\Delta\theta + \Delta r\Delta\theta$ and we ignore the second order term $\Delta r\Delta\theta$ giving $r\Delta\theta$.

So:

$$\Delta f = \left(\hat{\mathbf{e}}_{\mathbf{r}} \frac{\partial f}{\partial r} + \hat{\mathbf{e}}_{\theta} \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\mathbf{e}}_{\mathbf{z}} \frac{\partial f}{\partial z}\right) \boldsymbol{.}\, \Delta\vec{s} + \text{ higher order terms} \tag{160}$$

Note the factor of $\frac{1}{r}$ which ensures on doing the dot product we get the right form of $\Delta f$.

But $\Delta \mathbf{s} = \hat{\mathbf{u}}\, \Delta s$ where $\hat{\mathbf{u}}$ is a unit vector in the direction of $\mathbf{s}$.

Therefore:

$$\lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} = \frac{df}{ds} = \underbrace{\left(\hat{\mathbf{e}}_{\mathbf{r}} \frac{\partial f}{\partial r} + \hat{\mathbf{e}}_{\theta} \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\mathbf{e}}_{\mathbf{z}} \frac{\partial f}{\partial z}\right)}_{\nabla f} \boldsymbol{.}\hat{\mathbf{u}} \tag{161}$$

Thus:

$$\nabla_{\text{cyl}} = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{162}$$

We now have all the ingredients to finish the recipe by calculating the Laplacian as follows (see (143) ):

$$\Delta\Phi = \nabla^2 \Phi = \nabla \boldsymbol{.}\, (\nabla\Phi) = \nabla \boldsymbol{.}\, \mathbf{F} = \text{div}\, \mathbf{F} \tag{163}$$

where $\nabla_{\text{cyl}}\Phi = \mathbf{F}$

i.e:

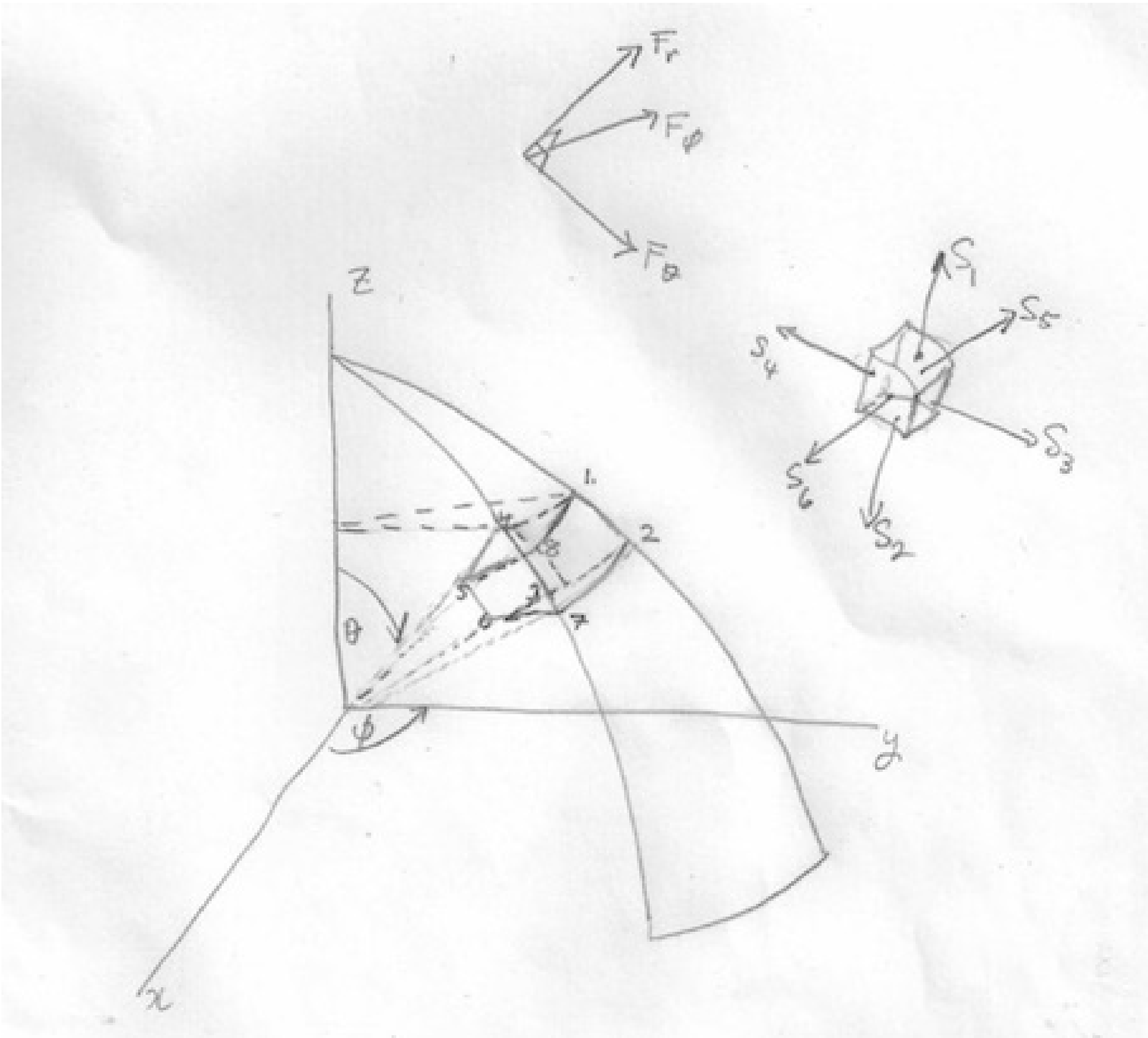
$$\begin{pmatrix} \frac{\partial \Phi}{\partial r} \\ \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \\ \frac{\partial \Phi}{\partial z} \end{pmatrix} = \begin{pmatrix} F_r \\ F_\theta \\ F_z \end{pmatrix} \tag{164}$$

Therefore using (124) and (164):

$$\begin{aligned} \Delta\Phi &= \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2} \frac{\partial}{\partial \theta}\left(\frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial z}\right) \\ &= \boxed{\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2}} \end{aligned} \tag{165}$$

# 7 Derivation of the Laplacian in spherical coordinates using gradient and divergence techniques

Using the same recipe we now calculate the Laplacian in spherical coordinates. This is more intricate because one has to be careful with the form of each elementary surface. In this representation $(r, \theta, \phi)$ note that some authors interchange the roles of $\theta$ and $\phi$ so that needs to be kept in mind when comparing the final form of the Laplacian.




The elementary volume is made up of radial and two angular elementary arcs as follows. To obtain the circular arc which is parallel to the $(x, y)$ plane (ie in the $\phi$ direction) the radius is $(r + \Delta r)\sin\theta$ so that the arc length is $(r + \Delta r)\sin\theta\,\Delta\phi = r\sin\theta\Delta\phi + \underbrace{\Delta r\Delta\phi\sin\theta}_{\text{ignore as second order}} \approx r\sin\theta\,\Delta\phi$. The radial dimension of the elementary volume has length $\Delta r$ and the arc length in the $\theta$ direction is:

$(r+\Delta r)\Delta\theta = r\Delta\theta + \underbrace{\Delta r\Delta\theta}_{\text{ignore as second order}} \approx r\Delta\theta$. Thus:

$$\Delta V = r\sin\theta\,\Delta\phi \times \Delta r \times r\Delta\theta = r^2\sin\theta\,\Delta r\,\Delta\phi\,\Delta\theta \tag{166}$$

For convenience the elementary surface areas are set out below. Each elementary surface is centred on $(r,\theta,\phi)$ and is used in working out the approximations to the relevant surface integrals:

$$\begin{aligned}
\Delta S_1 =& (r+\frac{\Delta r}{2})\ \sin\theta\Delta\phi \times (r+\frac{\Delta r}{2})\Delta\theta = (r+\frac{\Delta r}{2})^2\sin\theta\,\Delta\phi\,\Delta\theta\\
\Delta S_2 =& (r-\frac{\Delta r}{2})\,\Delta\theta \times (r-\frac{\Delta r}{2})\ \sin\theta\,\Delta\phi = (r-\frac{\Delta r}{2})^2\sin\theta\,\Delta\phi\,\Delta\theta\\
\Delta S_3 =& (r+\frac{\Delta r}{2})\ \sin(\theta+\frac{\Delta\theta}{2})\,\Delta r\,\Delta\phi\\
\Delta S_4 =& (r+\frac{\Delta r}{2})\ \sin\theta\,\Delta r\,\Delta\phi\\
\Delta S_5 =& (r+\frac{\Delta r}{2})\,\Delta r\,\Delta\theta\\
\Delta S_6 =& (r+\frac{\Delta r}{2})\,\Delta r\,\Delta\theta
\end{aligned} \tag{167}$$

$$\begin{aligned}
\iint_{S_1} \mathbf{F}\,.\,\hat{\mathbf{n}}\,dS =& \iint_{S_1} F_r\,dS\\
\approx& F_r\Big(r+\frac{\Delta r}{2},\theta,\phi\Big)\Delta S_1\\
=& F_r\Big(r+\frac{\Delta r}{2},\theta,\phi\Big)\,(r+\frac{\Delta r}{2})^2\sin\theta\,\Delta\phi\,\Delta\theta
\end{aligned} \tag{168}$$

$$\begin{aligned}
\iint_{S_2} \mathbf{F}\,.\,\hat{\mathbf{n}}\,dS =& -\iint_{S_2} F_r\,dS\\
\approx& -F_r\Big(r-\frac{\Delta r}{2},\theta,\phi\Big)\Delta S_2\\
=& -F_r\Big(r-\frac{\Delta r}{2},\theta,\phi\Big)\,(r-\frac{\Delta r}{2})^2\sin\theta\,\Delta\phi\,\Delta\theta
\end{aligned} \tag{169}$$

Therefore:

$$\begin{aligned}
\lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{S_1+S_2} \mathbf{F}\,.\,\hat{\mathbf{n}}\,dS \approx& \frac{1}{r^2\sin\theta\Delta r\Delta\phi\Delta\theta}\Big[F_r\Big(r+\frac{\Delta r}{2},\theta,\phi\Big)\,(r+\frac{\Delta r}{2})^2\sin\theta\,\Delta\phi\,\Delta\theta -\\
&F_r\Big(r-\frac{\Delta r}{2},\theta,\phi\Big)\,(r-\frac{\Delta r}{2})^2\sin\theta\,\Delta\phi\,\Delta\theta\Big]\\
=& \lim_{\Delta r\to 0}\frac{1}{r^2}\left[\frac{F_r\Big(r+\frac{\Delta r}{2},\theta,\phi\Big)\,(r+\frac{\Delta r}{2})^2 - F_r\Big(r-\frac{\Delta r}{2},\theta,\phi\Big)\,(r-\frac{\Delta r}{2})^2}{\Delta r}\right]\\
\to& \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2F_r\Big)
\end{aligned} \tag{170}$$

$$\begin{aligned}\iint_{S_3} \mathbf{F}\,.\,\hat{\mathbf{n}}\,dS &= \iint_{S_3} F_\theta\, dS \\ &\approx F_\theta\Big(r, \theta + \frac{\Delta\theta}{2}, \phi\Big)\Delta S_3 \\ &= F_\theta\Big(r, \theta + \frac{\Delta\theta}{2}, \phi\Big)\,(r + \frac{\Delta r}{2})\, \sin(\theta + \frac{\Delta\theta}{2})\,\Delta r\,\Delta\phi\end{aligned} \tag{171}$$

$$\begin{aligned}\iint_{S_4} \mathbf{F}\,.\,\hat{\mathbf{n}}\,dS &= -\iint_{S_4} F_\theta\, dS \\ &\approx -F_\theta\Big(r, \theta - \frac{\Delta\theta}{2}, \phi\Big)\Delta S_4 \\ &= -F_\theta\Big(r, \theta - \frac{\Delta\theta}{2}, \phi\Big)\,(r + \frac{\Delta r}{2})\, \sin\theta\,\Delta r\,\Delta\phi\end{aligned} \tag{172}$$

$$\begin{aligned}&\lim_{\Delta V\to 0} \frac{1}{\Delta V}\iint_{S_3+S_4} \mathbf{F}\,.\,\hat{\mathbf{n}}\,dS \\ &\approx \frac{1}{r^2\sin\theta\Delta r\Delta\phi\Delta\theta}\Big[F_\theta\Big(r, \theta + \frac{\Delta\theta}{2}, \phi\Big)\,(r + \frac{\Delta r}{2})\, \sin(\theta + \frac{\Delta\theta}{2})\,\Delta r\,\Delta\phi \\ &-F_\theta\Big(r, \theta - \frac{\Delta\theta}{2}, \phi\Big)\,(r + \frac{\Delta r}{2})\, \sin\theta\,\Delta r\,\Delta\phi\Big] \\ &= \lim_{\Delta r\to 0}\frac{1}{r^2\sin\theta}\left[\frac{F_\theta\Big(r, \theta + \frac{\Delta\theta}{2}, \phi\Big)\,(r + \frac{\Delta r}{2})\, \sin(\theta + \frac{\Delta\theta}{2}) - F_\theta\Big(r, \theta - \frac{\Delta\theta}{2}, \phi\Big)\,(r + \frac{\Delta r}{2})\, \sin\theta}{\Delta\theta}\right] \\ &\to \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\, F_\theta\Big)\end{aligned} \tag{173}$$

$$\begin{aligned}\iint_{S_5} \mathbf{F}\,.\,\hat{\mathbf{n}}\,dS &= \iint_{S_5} F_\phi\, dS \\ &\approx F_\phi\Big(r, \theta, \phi + \frac{\Delta\phi}{2}\Big)\Delta S_5 \\ &= F_\phi\Big(r, \theta, \phi + \frac{\Delta\phi}{2}\Big)\,(r + \frac{\Delta r}{2})\,\Delta r\,\Delta\theta\end{aligned} \tag{174}$$

$$\begin{aligned}\iint_{S_6} \mathbf{F}\,.\,\hat{\mathbf{n}}\,dS &= -\iint_{S_6} F_\phi\, dS \\ &\approx -F_\phi\Big(r, \theta, \phi - \frac{\Delta\Phi}{2}\Big)\Delta_{S_6} \\ &= -F_\phi\Big(r, \theta, \phi - \frac{\Delta\phi}{2}\Big)\,(r + \frac{\Delta r}{2})\,\Delta r\,\Delta\theta\end{aligned} \tag{175}$$

Therefore:

$$
\begin{aligned}
\lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{S_5+S_6} \mathbf{F} \,.\, \hat{\mathbf{n}} \, dS &= \frac{1}{r^2 \sin\theta \Delta r \Delta\phi \Delta\theta} \iint_{S_5+S_6} F_\phi \, dS \\
&\approx \frac{1}{r^2 \sin\theta \Delta r \Delta\phi \Delta\theta} \iint_{S_5+S_6} F_\phi \, \Delta S_6 \\
&= \frac{1}{r^2 \sin\theta \Delta r \Delta\phi \Delta\theta} \Big[ F_\phi\Big(r, \theta, \phi + \frac{\Delta\phi}{2}\Big)(r + \frac{\Delta r}{2}) \, \Delta r \, \Delta\theta \\
&\quad - F_\phi\Big(r, \theta, \phi - \frac{\Delta\phi}{2}\Big)(r + \frac{\Delta r}{2}) \, \Delta r \, \Delta\theta \Big] \\
&= \frac{1}{r^2 \sin\theta} \left[ \frac{F_\phi\Big(r, \theta, \phi + \frac{\Delta\phi}{2}\Big)(r + \frac{\Delta r}{2}) - F_\phi\Big(r, \theta, \phi - \frac{\Delta\phi}{2}\Big)(r + \frac{\Delta r}{2})}{\Delta\phi} \right] \\
&\to \frac{1}{r \sin\theta} \frac{\partial F_\phi}{\partial \phi}
\end{aligned} \tag{176}
$$

Putting it all together we have:

$$
\boxed{\operatorname{div} \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r}\Big(r^2 F_r\Big) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}\Big(\sin\theta \, F_\theta\Big) + \frac{1}{r \sin\theta} \frac{\partial F_\phi}{\partial \phi}} \tag{177}
$$

As before we need to work ou the gradient and as a first step we need to determine the elementary vectorial displacement $\Delta\mathbf{s}$. The displacements in the 3 directions are (see the diagram above):

radial direction: $\Delta r$

$\theta$ direction: $(r + \Delta r)\Delta\theta = r\Delta\theta + \underbrace{\Delta r \Delta\theta}_{\text{second order}} \approx r\Delta\theta$

$\phi$ direction: $(r + \Delta r)\, \sin\theta \, \Delta\phi = r \sin\theta \, \Delta\phi + \underbrace{\sin\theta \, \Delta r \Delta\phi}_{\text{second order}} \approx r \sin\theta \, \Delta\phi$

Thus in terms of its natural basis the vectorial displacement is:

$$
\Delta\mathbf{s} = \hat{\mathbf{e}}_\mathbf{r} \, \Delta r + \hat{\mathbf{e}}_\theta \, r\Delta\theta + \hat{\mathbf{e}}_\phi \, r \sin\theta \, \Delta\phi \tag{178}
$$

We also have:

$$
\Delta f = \frac{\partial f}{\partial r} \Delta r + \frac{\partial \theta}{\partial \theta} \Delta\theta + \frac{\partial f}{\partial \phi} \Delta\phi + \text{higher order terms} \tag{179}
$$

Thus we can write:

$$
\Delta f = \Big(\hat{\mathbf{e}}_\mathbf{r} \frac{\partial f}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \hat{\mathbf{e}}_\phi \, \frac{1}{r \, \sin\theta} \frac{\partial f}{\partial \phi}\Big) \,.\, \Delta\mathbf{s} + \text{ higher order terms} \tag{180}
$$

But $\Delta\mathbf{s} = \hat{\mathbf{u}} \, \Delta s$ where $\hat{\mathbf{u}}$ is a unit vector in the direction of $\mathbf{s}$.

Therefore:

$$\lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s} = \frac{df}{ds} = \underbrace{\left(\mathbf{\hat{e}_r} \frac{\partial f}{\partial r} + \mathbf{\hat{e}}_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + \mathbf{\hat{e}}_\phi \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi}\right)}_{\nabla f} \mathbf{.\hat{u}} \tag{181}$$

Thus:

$$\nabla_{\text{sph}} = \begin{pmatrix} \frac{\partial}{\partial r} \\ \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \\ \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \end{pmatrix} \tag{182}$$

As before we now have all the ingredients to finish the recipe by calculating the Laplacian as follows (I use $\Phi$ as the function symbol while $\phi$ is the angular variable ):

$$\Delta\Phi = \nabla^2\Phi = \nabla \,\mathbf{.}\, (\nabla\Phi) = \nabla \,\mathbf{.}\, \mathbf{F} = \text{div}\, \mathbf{F} \tag{183}$$

where $\nabla_{\text{sph}}\Phi = \mathbf{F}$

i.e:

$$\begin{pmatrix} \frac{\partial \Phi}{\partial r} \\ \\ \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \\ \\ \frac{1}{r \sin\theta} \frac{\partial \Phi}{\partial \phi} \end{pmatrix} = \begin{pmatrix} F_r \\ \\ F_\theta \\ \\ F_\phi \end{pmatrix} \tag{184}$$

Therefore using (177):

$$\boxed{\Delta\Phi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2}} \tag{185}$$

Note that if $\theta$ and $\phi$ are interchanged (as some authors do ) the form of the Laplacian will be:

$$\boxed{\Delta\Phi = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\phi} \frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial \Phi}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi} \frac{\partial^2 \Phi}{\partial \theta^2}} \tag{186}$$

The Laplacian (185) can be written in the alternative form:

$$\Delta\Phi = \frac{1}{r} \frac{\partial^2}{\partial r^2}\left(r\Phi\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2} \tag{187}$$

That the two forms are equivalent is based on this equality:

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = \frac{1}{r} \frac{\partial^2}{\partial r^2}\left(r\Phi\right) \tag{188}$$

We can get the gradient in spherical coordinates quite efficiently using the following approach. In the $(r, \theta, \phi)$ coordinates a general point $\mathbf{x}$ has the following representation. See the diagram at the beginning of this section:

$$\mathbf{x} = \begin{pmatrix} r \sin\theta \, \cos\phi \\ r \sin\theta \, \sin\phi \\ r \cos\theta \end{pmatrix} \tag{189}$$

Then:

$$d\mathbf{x} = \begin{pmatrix} \sin\theta \, \cos\phi \\ \sin\theta \, \sin\phi \\ \cos\theta \end{pmatrix} dr + r \begin{pmatrix} \cos\theta \, \cos\phi \\ \cos\theta \, \sin\phi \\ -\sin\theta \end{pmatrix} d\theta + r \begin{pmatrix} -\sin\theta \, \sin\phi \\ \sin\theta \, \cos\phi \\ 0 \end{pmatrix} d\phi \tag{190}$$

We can write (190) as follows in terms of unit vectors $\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\phi$:

$$d\mathbf{x} = \mathbf{e_r} \, dr + r \, \mathbf{e}_\theta \, d\theta + r \, \sin\theta \, \mathbf{e}_\phi \, d\phi \tag{191}$$

where:

$$\mathbf{e_r} = \begin{pmatrix} \sin\theta \, \cos\phi \\ \sin\theta \, \sin\phi \\ \cos\theta \end{pmatrix} \tag{192}$$

$$\mathbf{e}_\theta = \begin{pmatrix} \cos\theta \, \cos\phi \\ \cos\theta \, \sin\phi \\ -\sin\theta \end{pmatrix} \tag{193}$$

$$\mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \tag{194}$$

It is easily verified that $\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\phi$ are unit vectors eg $|\mathbf{e_r}| = \sqrt{\sin^2\theta \, \cos^2\phi + \sin^2\theta \, \sin^2\phi + \cos^2\theta} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$.

The gradient of $\Phi$ is:

$$\nabla\Phi = \frac{\partial\Phi}{\partial r} \, dr + \frac{\partial\Phi}{\partial\theta} \, d\theta + \frac{\partial\Phi}{\partial\phi} \, d\phi \tag{195}$$

But:

$$d\Phi = \nabla\Phi \, . \, d\mathbf{x} \tag{196}$$

So using (191) we have:

$$
\begin{aligned}
\frac{\partial \Phi}{\partial r}\,dr + \frac{\partial \Phi}{\partial \theta}\,d\theta + \frac{\partial \Phi}{\partial \phi}\,d\phi =& \nabla\Phi \,\boldsymbol{.}\, \left(\mathbf{e_r}\,dr + r\,\mathbf{e}_\theta\,d\theta + r\,\sin\theta\,\mathbf{e}_\phi\,d\phi\right) \\
=& \nabla\Phi \,\boldsymbol{.}\, \mathbf{e_r}\,dr + \nabla\Phi \,\boldsymbol{.}\, \mathbf{e}_\theta\,r\,d\theta + \nabla\Phi \,\boldsymbol{.}\, \mathbf{e}_\phi\,r\sin\theta\,d\phi
\end{aligned}
\tag{197}
$$

Equating coefficients we have:

$$
\begin{aligned}
\frac{\partial \Phi}{\partial r} =& \nabla\Phi \,\boldsymbol{.}\, \mathbf{e_r} \\
\frac{1}{r}\frac{\partial \Phi}{\partial \theta} =& \nabla\Phi \,\boldsymbol{.}\, \mathbf{e}_\theta \\
\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi} =& \nabla\Phi \,\boldsymbol{.}\, \mathbf{e}_\phi
\end{aligned}
\tag{198}
$$

The LHS of (198) comprise the projections of the vector $\nabla\Phi$ onto the respective unit vectors. Therefore:

$$
\begin{aligned}
\nabla\Phi =& (\nabla\Phi \,\boldsymbol{.}\, \mathbf{e_r})\,\mathbf{e_r} + (\nabla\Phi \,\boldsymbol{.}\, \mathbf{e}_\theta)\,\mathbf{e}_\theta + (\nabla\Phi \,\boldsymbol{.}\, \mathbf{e}_\phi)\,\mathbf{e}_\phi \\
=& \frac{\partial \Phi}{\partial r}\,\mathbf{e_r} + \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\,\mathbf{e}_\phi
\end{aligned}
\tag{199}
$$

Thus, as before, the gradient operator has the following form in spherical coordinates:

$$
\nabla_{\text{sph}} = \begin{pmatrix} \frac{\partial}{\partial r} \\ \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \\ \frac{1}{r\,\sin\theta}\frac{\partial}{\partial \phi} \end{pmatrix}
\tag{200}
$$

One can apply the same logic to the gradient in cylindrical coordinates.

## 8 A more general approach to the Laplacian

The final approach I will look at revolves around a general expression for the Laplacian in orthogonal curvilinear coordinates. This is the most powerful method because all the "usual suspects" (Cartesian, polar, cylindrical, spherical etc) are all contained within the general formula. Even more generally this procedure can comprehend non-orthogonal systems, although I will not deal with that possibility. The downside is that there is some differential geometry and if you serious, tensor theory.

So let's begin. We assume we have an orthogonal curvilinear coordinate system. If you have surfaces $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ where the $c_i$ are constants, each pair of the surfaces will intersect in a curvilinear coordinate curve and if they intersect at right angles we will have an orthogonal system.

We now have to relate the position vector of a point in Cartesian coordinates to these new coordinates. The relationship is something of this form:

$$
\begin{aligned}
x &= x(u_1, u_2, u_3) \\
y &= y(u_1, u_2, u_3) \\
z &= z(u_1, u_2, u_3)
\end{aligned} \tag{201}
$$

For instance in cylindrical coordinates $(r, \phi, z)$ we have:

$$
\begin{aligned}
x &= r\cos\phi \\
y &= r\sin\phi \\
z &= z
\end{aligned} \tag{202}
$$

The position vector of a point P is :

$$
\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{203}
$$

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard Cartesian unit vectors.

The crux of this approach is to obtain a representation of position vectors in terms of tangential (contravariant) unit base vectors and normal (covariant) unit base vectors. Thus the relevant vector is represented in either tangential or normal components. Once this is understood the mechanics are relatively straightforward.

A tangent vector to $u_1$ at P is simply:

$$
\frac{\partial \mathbf{r}}{\partial u_1} \tag{204}
$$

and a unit vector in that direction is just:

$$
\mathbf{e_1} = \frac{\frac{\partial \mathbf{r}}{\partial u_1}}{\left|\frac{\partial \mathbf{r}}{\partial u_1}\right|} \tag{205}
$$

So we can write the tangent vector in terms of a scale factor $h_1$ as follows:

$$
\frac{\partial \mathbf{r}}{\partial u_1} = h_1 \mathbf{e_1} \tag{206}
$$

where $h_1 = \left|\frac{\partial \mathbf{r}}{\partial u_1}\right|$

Replicating that logic we have for $i = 1, 2, 3$:

$$
\frac{\partial \mathbf{r}}{\partial u_i} = h_i \mathbf{e_i} \tag{207}
$$

where $h_i = \left|\frac{\partial \mathbf{r}}{\partial u_i}\right|$

Now for the normal representation. We know from basic calculus that a vector which is normal to the surface $u_1 = c_1$ at P is:

$$\nabla u_1 \tag{208}$$

Hence a unit vector in this direction is:

$$\mathbf{E_1} = \frac{\nabla u_1}{|\nabla u_1|} \tag{209}$$

As before we have for $i = 1, 2, 3$:

$$\mathbf{E_i} = \frac{\nabla u_i}{|\nabla u_i|} \tag{210}$$

The $\frac{\partial \mathbf{r}}{\partial u_i}$ and $\nabla u_i$ are a reciprocal system of vectors in the sense that:

$$\frac{\partial \mathbf{r}}{\partial u_i} \cdot \nabla u_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{211}$$

This is seen as follows: Since:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u_i} du_i \tag{212}$$

when we take the dot product with $\nabla u_1$ (which is orthogonal to $\frac{\partial \mathbf{r}}{\partial u_1}$ ) we get:

$$\nabla u_1 \cdot d\mathbf{r} = du_1 = \sum_{i=1}^{3} \left( \nabla u_i \cdot \frac{\partial \mathbf{r}}{\partial u_i} \right) du_i \tag{213}$$

Hence:

$$\begin{aligned} \nabla u_1 \cdot \frac{\partial \mathbf{r}}{\partial u_1} &= 1 \\ \nabla u_1 \cdot \frac{\partial \mathbf{r}}{\partial u_2} &= 0 \\ \nabla u_1 \cdot \frac{\partial \mathbf{r}}{\partial u_3} &= 0 \end{aligned} \tag{214}$$

Similarly for $\nabla u_2$ and $\nabla u_3$.

To get a general expression for the gradient in orthogonal curvilinear coordinates we proceed as follows. We want $\nabla \Phi$:

$$\nabla \Phi = \sum_{i=1}^{3} f_i \, \mathbf{e_i} \tag{215}$$

$$
\begin{aligned}
d\mathbf{r} &= \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u_i}\, du_i \\
&= \sum_{i=1}^{3} h_i\, \mathbf{e_i}\, du_i \quad \text{using (181)}
\end{aligned} \tag{216}
$$

But the directional derivative is:

$$
\begin{aligned}
d\Phi &= \nabla\Phi \,.\, d\mathbf{r} \\
&= \sum_{i=1}^{3} h_i\, f_i\, du_i \\
&= \sum_{i=1}^{3} \frac{\partial \Phi}{\partial u_i}\, du_i
\end{aligned} \tag{217}
$$

Equating coefficients in the last two lines of (191) we have for $i = 1, 2, 3$:

$$
f_i = \frac{1}{h_i} \frac{\partial \Phi}{\partial u_i} \tag{218}
$$

Hence:

$$
\boxed{\nabla\Phi = \sum_{i=1}^{3} \frac{\mathbf{e_i}}{h_i} \frac{\partial \Phi}{\partial u_i}} \tag{219}
$$

We can also write (219) in operator form:

$$
\boxed{\nabla = \sum_{i=1}^{3} \frac{\mathbf{e_i}}{h_i} \frac{\partial}{\partial u_i}} \tag{220}
$$

Note that in the Cartesian coordinate system $h_1 = h_2 = h_3 = 1$ and so we can identify $\mathbf{e_1}$ as $\mathbf{i}$ , $\mathbf{e_2}$ as $\mathbf{j}$ and $\mathbf{e_3}$ as $\mathbf{k}$. This also means that the vectors $\mathbf{e_i}$ for $i = 1, 2, 3$ have the same cyclic behaviour as $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ie

$$
\begin{aligned}
\mathbf{e_1} \times \mathbf{e_2} &= \mathbf{e_3} \\
\mathbf{e_2} \times \mathbf{e_3} &= \mathbf{e_1} \\
\mathbf{e_3} \times \mathbf{e_1} &= \mathbf{e_2}
\end{aligned} \tag{221}
$$

In (219) let $\Phi = u_i$ then:

$$\nabla u_i = \frac{\mathbf{e_i}}{h_i} \implies |\nabla u_i| = \frac{|\mathbf{e_i}|}{h_i} = \frac{1}{h_i} \tag{222}$$

Because we want an expression for $\nabla^2\Phi = \nabla\,.\,\nabla\Phi$ we need a general expression for $\nabla\,.\,\mathbf{A} = \text{div}\,\mathbf{A}$. It turns out (as is shown below) that:

$$\nabla\,.\,(A_1\mathbf{e_1}) = \nabla\,.\,(A_1 h_2 h_3\,\nabla u_2 \times \nabla u_3) \tag{223}$$

with similar expressions for other components.

To get the cross product in (223) we use (221-222):

$$\nabla u_2 \times \nabla u_3 = \frac{\mathbf{e_2} \times \mathbf{e_3}}{h_2\,3} = \frac{\mathbf{e_1}}{h_2\,h_3} \tag{224}$$

Therefore:

$$\mathbf{e_1} = h_2 h_3\,\nabla u_2 \times \nabla u_3 \tag{225}$$

Similiar expressions follow for $\mathbf{e_2}$ and $\mathbf{e_3}$ ie

$$\begin{aligned} h_3 h_1\,\nabla u_3 \times \nabla u_1 &= \mathbf{e_2} \\ h_1 h_2\,\nabla u_1 \times \nabla u_2 &= \mathbf{e_3} \end{aligned} \tag{226}$$

The basic result we need is this. For a vector $\mathbf{B}$ and differentiable scalar function $\Phi$ the following holds:

$$\boxed{\nabla\,.\,(\Phi\mathbf{B}) = (\nabla\Phi)\,.\,\mathbf{B} + \Phi\,(\nabla\,.\,\mathbf{B})} \tag{227}$$

(227) is proved in the Appendix.

Now applying (227) to (223) with $\Phi = A_1 h_2 h_3$ and $\mathbf{B} = \nabla u_2 \times \nabla u_3$ we have:

$$\begin{aligned} \nabla\,.\,(A_1\mathbf{e_1}) &= \nabla\,.\,(A_1 h_2 h_3\,\nabla u_2 \times \nabla u_3) \\ &= \nabla(A_1 h_2 h_3)\,.\,\nabla u_2 \times \nabla u_3 + A_1 h_2 h_3\,\underbrace{\nabla\,.\,(\nabla u_2 \times \nabla u_3)}_{=0} \\ &= \nabla(A_1 h_2 h_3)\,.\,\frac{\mathbf{e_2}}{h_2} \times \frac{\mathbf{e_3}}{h_3} \\ &= \nabla(A_1 h_2 h_3)\,.\,\frac{\mathbf{e_1}}{h_2 h_3} \\ &= \Big[\frac{\mathbf{e_1}}{h_1}\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\mathbf{e_2}}{h_2}\frac{\partial}{\partial u_2}(A_1 h_2 h_3) + \frac{\mathbf{e_3}}{h_3}\frac{\partial}{\partial u_3}(A_1 h_2 h_3)\Big]\,.\,\frac{\mathbf{e_1}}{h_2 h_3} \qquad \text{using (220)} \\ &= \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(A_1 h_2 h_3) \end{aligned} \tag{228}$$

Note that $\nabla \cdot (\nabla u_2 \times \nabla u_3) = 0$ because of the following:

$$
\begin{aligned}
\nabla \cdot (\nabla u_2 \times \nabla u_3) =& \nabla \cdot \frac{\mathbf{e_1}}{h_2 h_3} \\
=& \Big( \sum_{i=1}^{3} \frac{\mathbf{e_i}}{h_i} \frac{\partial}{\partial u_i} \Big) \cdot \frac{\mathbf{e_1}}{h_2 h_3} \\
=& 0
\end{aligned} \tag{229}
$$

We now repeat the process for the other factors:

$$
\begin{aligned}
\nabla \cdot (A_2 \mathbf{e_2}) =& \nabla \cdot (A_2 h_3 h_1 \, \nabla u_3 \times \nabla u_1) \\
=& \nabla (A_2 h_3 h_1) \cdot \nabla u_3 \times \nabla u_1 + A_2 h_3 h_1 \underbrace{\nabla \cdot (\nabla u_3 \times \nabla u_1)}_{=0} \\
=& \nabla (A_2 h_3 h_1) \cdot \frac{\mathbf{e_3}}{h_3} \times \frac{\mathbf{e_1}}{h_1} \\
=& \nabla (A_2 h_3 h_1) \cdot \frac{\mathbf{e_2}}{h_3 h_1} \\
=& \Big[ \frac{\mathbf{e_1}}{h_1} \frac{\partial}{\partial u_1} (A_2 h_3 h_1) + \frac{\mathbf{e_2}}{h_2} \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\mathbf{e_3}}{h_3} \frac{\partial}{\partial u_3} (A_2 h_3 h_1) \Big] \cdot \frac{\mathbf{e_2}}{h_3 h_1} \qquad \text{using (220)} \\
=& \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_2} (A_2 h_3 h_1)
\end{aligned} \tag{230}
$$

$$
\begin{aligned}
\nabla \cdot (A_3 \mathbf{e_3}) =& \nabla \cdot (A_3 h_1 h_2 \, \nabla u_1 \times \nabla u_2) \\
=& \nabla (A_3 h_1 h_2) \cdot \nabla u_1 \times \nabla u_2 + A_3 h_1 h_2 \underbrace{\nabla \cdot (\nabla u_1 \times \nabla u_2)}_{=0} \\
=& \nabla (A_3 h_1 h_2) \cdot \frac{\mathbf{e_1}}{h_1} \times \frac{\mathbf{e_2}}{h_2} \\
=& \nabla (A_3 h_1 h_2) \cdot \frac{\mathbf{e_3}}{h_1 h_2} \\
=& \Big[ \frac{\mathbf{e_1}}{h_1} \frac{\partial}{\partial u_1} (A_3 h_1 h_2) + \frac{\mathbf{e_2}}{h_2} \frac{\partial}{\partial u_2} (A_3 h_1 h_2) + \frac{\mathbf{e_3}}{h_3} \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \Big] \cdot \frac{\mathbf{e_3}}{h_1 h_2} \qquad \text{using (220)} \\
=& \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_3} (A_3 h_1 h_2)
\end{aligned} \tag{231}
$$

Putting it all together we see that:

$$
\boxed{
\begin{aligned}
\nabla \cdot \mathbf{A} =& \nabla \cdot (A_1 \mathbf{e_1} + A_2 \mathbf{e_2} + A_3 \mathbf{e_3}) \\
\nabla \cdot \mathbf{A} =& \frac{1}{h_1 h_2 h_3} \Big[ \frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \Big]
\end{aligned}
} \tag{232}
$$

We are now in a position to get the general form of the Laplacian in orthogonal curvilinear coordinates. If we let $\mathbf{A} = \nabla \Phi$ then $A_i = \frac{1}{h_i} \frac{\partial \Phi}{\partial u_i}$ and so by (232):

$$
\boxed{\begin{aligned}
\Delta =& \nabla \cdot \nabla\Phi \\
=& \nabla^2\Phi \\
=& \frac{1}{h_1h_2h_3}\Big[\frac{\partial}{\partial u_1}\Big(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\Big) + \frac{\partial}{\partial u_3}\Big(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\Big)\Big]
\end{aligned}} \tag{233}
$$

How do you remember the scale factors $h_1, h_2, h_3$ ? The easiest way is to remember what the elementary value components are in each system. The volume element in orthogonal curvilinear coordinates $u_1, u_2, u_3$ is $dV = h_1h_2h_3\, du_1du_2du_3$. For instance in Cartesian coordinates the elementary volume is $dx \times dy \times dz$ so $h_1 = h_2 = h_3 = 1$.

In cylindrical coordinates the elementary volume is ( see the diagram associated with (144) ):

$$
dV = (dr)\,(rd\theta)\,(dz) \tag{234}
$$

Note here that it is critical to properly associate the relevant factors with their basis vectors ie $h_1du_1 = 1 \times dr$, $h_2du_2 = r \times d\theta$ and $h_3du_3 = 1 \times dz$. Hence in this case $h_1 = 1$, $h_2 = r$ and $h_3 = 1$.

Plugging these values into (233) we get:

$$
\begin{aligned}
\Delta\Phi =& \frac{1}{r}\Big[\frac{\partial}{\partial r}\Big(r\frac{\partial\Phi}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\Big) + \frac{\partial}{\partial z}\Big(r\frac{\partial\Phi}{\partial z}\Big)\Big] \\
=& \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned} \tag{235}
$$

which is thankfully the same as (165).

For spherical coordinates the elementary volume is (see (166) and the diagram and discussion associated with it):

$$
dV = (dr)\,(rd\theta)\,(r\sin\theta\, d\phi) \tag{236}
$$

Hence $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin\theta$.

Plugging these values into (233) we get:

$$
\begin{aligned}
\Delta\Phi =& \frac{1}{r^2\sin\theta}\Big[\frac{\partial}{\partial r}\Big(r^2\sin\theta\frac{\partial\Phi}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\Phi}{\partial\theta}\Big) + \frac{\partial}{\partial\phi}\Big(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\Big)\Big] \\
=& \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2\,\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\Phi}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned} \tag{237}
$$

which is thankfully the same as (185).

The structure of (233) is actually easy to remember once you note how the scale factors figure in each derivative and it is easy to work out the elementary volumes by drawing a diagram in order to get the scale factors. Note the cyclic symmetry of the factors in each of the partial derivatives. That is for $i = 1$, the relevant fraction in the derivative has $h_1$ on the bottom and $h_2\, h_3$ on top and for $i = 2$ it cycles $2 \to 3 \to 1$ and for $i = 3$ it cycles $3 \to 1 \to 2$.

Can we use (233) to work out the Laplacian in polar coordinates $(r, \theta)$? Yes we can as long as we realise that in polar coordinates the "volume" element is a "degenerate" one - it is an area which we will still call $dV$:

$$dV = dr \times r\, d\theta \tag{238}$$

So $h_1 = 1$ and $h_2 = r$. Our expression for the Laplacian therefore has only two terms:

$$\Delta\Phi = \frac{1}{h_1 h_2}\Big[\frac{\partial}{\partial u_1}\Big(\frac{h_2}{h_1}\frac{\partial\Phi}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\Big)\Big] \tag{239}$$

So plugging the factors in we get:

$$\begin{aligned}\Delta\Phi =& \frac{1}{r}\Big[\frac{\partial}{\partial r}\Big(r\,\frac{\partial\Phi}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\Big)\Big] \\ =& \frac{1}{r}\Big[r\,\frac{\partial^2\Phi}{\partial r^2} + \frac{\partial\Phi}{\partial r} + \frac{1}{r}\,\frac{\partial^2\Phi}{\partial\theta^2}\Big] \\ =& \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\Phi}{\partial\theta^2}\end{aligned} \tag{240}$$

which is the same as (45).

# 9 The final generalisation - the tensor form of the Laplacian

The expression in (233) can be generalised even further using the techniques of tensor calculus. I will only state what the Laplacian looks like first (see [5], page 231 ):

$$\boxed{\Delta = \nabla^2\Phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\Big(\sqrt{g}\,g^{kr}\frac{\partial\Phi}{\partial x^r}\Big)} \tag{241}$$

To prove (241) is a major undertaking which involves knowing how to perform covariant differentiation and several other things. This is already a long paper and I am not going to double its length by a full blown account of tensor calculus). In summary one starts with an expression for the divergence:

$$\text{div}\, A^p = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\,A^k) \tag{242}$$

The gradient of $\Phi$ is grad $\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial x^r}$ which is defined as a covariant derivative of $\Phi$ often written as $\Phi_{,r}$. The rank 1 contravariant tensor associated with $\Phi_{,r}$ is $A^k = g^{kr}\frac{\partial\Phi}{\partial x^r}$. Then:

$$\nabla^2\Phi = \text{div}\,(\nabla\Phi) = \text{div}\,(g^{kr}\frac{\partial\Phi}{\partial x^r}) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\,g^{kr}\frac{\partial\Phi}{\partial x^r}) \tag{243}$$

There are some things of a general nature to note about (241). First the determinant of the metric tensor, that is $g$, takes the place of $h_1h_2h_3$ which makes sense because of the following. If the derivative of the position vector is $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u_1}\, du_1 + \frac{\partial \mathbf{r}}{\partial u_2}\, du_2 + \frac{\partial \mathbf{r}}{\partial u_3}\, du_3$ then:

$$ds^2 = \sum_{p=1}^{3}\sum_{q=1}^{3} g_{pq}\, du_p du_q \tag{244}$$

or simply $g_{pq}\, du_p du_q$ using the Einstein summation convention. Note that by $ds^2$ we mean $(ds)^2$ rather than some contravariant index.

Thus:

$$g_{pq} = \frac{\partial \mathbf{r}}{\partial u_p} \cdot \frac{\partial \mathbf{r}}{\partial u_q} \tag{245}$$

In orthogonal coordinate systems under consideration in this paper $g_{pq} = 0$ when $p \neq q$. Recall that in orthogonal curvilinear coordinates the elementary volume element is:

$$dV = |(h_1 du_1 \mathbf{e_1}) \cdot (h_2 du_2 \mathbf{e_2} \times (h_3 du_3 \mathbf{e_3})| = h_1 h_2 h_3\, du_1 du_2 du_3 \tag{246}$$

where the $\mathbf{e_k} = \frac{\frac{\partial \mathbf{r}}{\partial u_k}}{\frac{1}{\left|\frac{\partial \mathbf{r}}{\partial u_k}\right|}}$. In the tensor representation the scale factors $h_1, h_2, h_3$ are replaced by $\sqrt{g}$ which in turn reflects the scale factors for the volume.

The other point to note that in (233) we have factors such as $\frac{h_2 h_3}{h_1}$ which are replaced by $\sqrt{g}\, g^{kr}$ where $g^{kr}$ is the conjugate metric tensor to $g_{kr}$. The relationship between the metric tensor and its conjugate boils down to this:

$$g_{jk}\, g^{pk} = \delta^p_j \tag{247}$$

where $\delta^p_j$ is the Kronecker delta ie $\delta^p_j = 1$ if $p = j$ and 0 if $p \neq j$. In matrix terms the conjugate metric tensor involves calcuating cofactors and this is why you get terms such as $\frac{h_2 h_3}{h_1}$ since you are multiplying a determinant of the metric tensor (ie $\sqrt{g}$ ) by a cofactor type of term.

## 9.1 Applying the tensor formula to cylindrical coordinates

For cylindrical coordinates we have:

$$\begin{aligned} x^1 &= r\cos\theta \\ x^2 &= r\sin\theta \\ x^3 &= z \end{aligned} \tag{248}$$

Hence:

$$
\begin{aligned}
dx^1 &= -r \sin\theta \, d\theta + \cos\theta \, dr \\
dx^2 &= r \cos\theta \, d\theta + \sin\theta \, dr \\
dx^3 &= dz
\end{aligned} \tag{249}
$$

The squared arc length is:

$$
ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \tag{250}
$$

You can now read off the metric tensor components (knowing that the off diagonal ones are 0):

$$
\begin{aligned}
g_{11} &= 1 \\
g_{22} &= r^2 \\
g_{33} &= 1 \\
g_{12} &= g_{21} = 0 \\
g_{23} &= g_{32} = 0 \\
g_{31} &= g_{13} = 0
\end{aligned} \tag{251}
$$

In matrix form the tensor looks like this:

$$
(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{252}
$$

The determinant of (252) is $g = r^2$ hence $\sqrt{g} = r$. Note that because of (247) we have:

$$
\begin{aligned}
g^{11} &= \frac{1}{g_{11}} \\
g^{22} &= \frac{1}{g_{22}} \\
g^{33} &= \frac{1}{g_{33}}
\end{aligned} \tag{253}
$$

In (241) we need the conjugate metric tensor $g^{kr}$ and this has a simple matrix form where the diagonal terms are inverted:

$$
(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{254}
$$

We now plug the relevant bits into (241) as follows:

$$
\begin{aligned}
\nabla^2\Phi &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\Big(\sqrt{g}\,g^{kr}\frac{\partial\Phi}{\partial x^r}\Big)\\
&= \frac{1}{r}\Big[\frac{\partial}{\partial r}\Big(r.1\frac{\partial\Phi}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(r.\frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}\Big) + \frac{\partial}{\partial z}\Big(r.1\frac{\partial\Phi}{\partial z}\Big)\Big]\\
&= \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}
\tag{255}
$$

which is the same as (165).

If we wanted to calculate the conjugate metric tensor using cofactors the approach is as follows (see (252) ):

$$
g^{11} = \frac{\text{cofactor of } g_{11}}{g} = \frac{1}{r^2}\begin{vmatrix} r^2 & 0 \\ 0 & 1 \end{vmatrix} = 1 \tag{256}
$$

$$
g^{22} = \frac{\text{cofactor of } g_{22}}{g} = \frac{1}{r^2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{r^2} \tag{257}
$$

$$
g^{33} = \frac{\text{cofactor of } g_{33}}{g} = \frac{1}{r^2}\begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix} = 1 \tag{258}
$$

## 9.2 Applying the tensor formula to spherical coordinates

For spherical coordinates we have:

$$
\begin{aligned}
x^1 &= r\,\sin\theta\,\cos\phi\\
x^2 &= r\sin\theta\,\sin\phi\\
x^3 &= r\,\cos\theta
\end{aligned}
\tag{259}
$$

Hence:

$$
\begin{aligned}
dx^1 &= \sin\theta\,\cos\phi\,dr + r\cos\theta\,\cos\phi\,d\theta - r\sin\theta\,\sin\phi\,d\phi\\
dx^2 &= \sin\theta\,\sin\phi\,dr + r\,\cos\theta\,\sin\phi\,d\theta + r\,\sin\theta\,\cos\phi\,d\phi\\
dx^3 &= \cos\theta\,dr - r\,\sin\theta\,d\theta
\end{aligned}
\tag{260}
$$

The squared arc length is:

$$
ds^2 = dr^2 + r^2 d\theta^2 + r^2\,\sin^2\theta\,d\phi^2 \tag{261}
$$

You can now read off the metric tensor components (knowing that the off diagonal ones are 0):

$$
\begin{aligned}
g_{11} &= 1 \\
g_{22} &= r^2 \\
g_{33} &= r^2 \ \sin^2\theta \\
g_{12} &= g_{21} = 0 \\
g_{23} &= g_{32} = 0 \\
g_{31} &= g_{13} = 0
\end{aligned}
\tag{262}
$$

In matrix form the tensor looks like this:

$$
(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \ \sin^2\theta \end{pmatrix} \tag{263}
$$

The determinant of (263) is $g = r^4 \ \sin^2\theta$ hence $\sqrt{g} = r^2 \ \sin\theta$. Note that because of (247) we have:

$$
(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \ \sin^2\theta} \end{pmatrix} \tag{264}
$$

We now plug the relevant bits into (241):

$$
\begin{aligned}
\nabla^2\Phi &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\Big( \sqrt{g}\, g^{kr} \frac{\partial \Phi}{\partial x^r}\Big) \\
&= \frac{1}{r^2 \sin\theta}\Big[ \frac{\partial}{\partial r}\Big(r^2 \sin\theta . 1 \frac{\partial\Phi}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(r^2 \sin\theta . \frac{1}{r^2} \frac{\partial\Phi}{\partial\theta}\Big) + \frac{\partial}{\partial\phi}\Big(r^2 \sin\theta . \frac{1}{r^2 \sin^2\theta} \frac{\partial\Phi}{\partial\phi}\Big)\Big] \\
&= \frac{1}{r^2} \frac{\partial}{\partial r}\Big(r^2 \frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\Big(\sin\theta \frac{\partial\Phi}{\partial\theta}\Big) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}
\tag{265}
$$

which is the same as (185).

If we wanted to calculate the conjugate metric tensor using cofactors the approach is as follows (see (263) ):

$$
g^{11} = \frac{\text{cofactor of } g_{11}}{g} = \frac{1}{r^4 \ \sin^2\theta} \begin{vmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{vmatrix} = 1 \tag{266}
$$

$$
g^{22} = \frac{\text{cofactor of } g_{22}}{g} = \frac{1}{r^4 \ \sin^2\theta} \begin{vmatrix} 1 & 0 \\ 0 & r^2 \sin^2\theta \end{vmatrix} = \frac{1}{r^2} \tag{267}
$$

$$
g^{33} = \frac{\text{cofactor of } g_{33}}{g} = \frac{1}{r^4 \ \sin^2\theta} \begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix} = \frac{1}{r^2 \sin^2\theta} \tag{268}
$$

### 9.3 Applying the tensor formula to polar coordinates

For polar coordinates we have:

$$
\begin{aligned}
x^1 &= r\cos\theta \\
x^2 &= r\sin\theta
\end{aligned}
\tag{269}
$$

Hence:

$$
\begin{aligned}
dx^1 &= \cos\theta\, dr - r\sin\theta\, d\theta \\
dx^2 &= \sin\theta\, dr + r\cos\theta\, d\theta
\end{aligned}
\tag{270}
$$

The squared arc length is:

$$
ds^2 = dr^2 + r^2 d\theta^2 \tag{271}
$$

You can now read off the metric tensor components (knowing that the off diagonal ones are 0):

$$
\begin{aligned}
g_{11} &= 1 \\
g_{22} &= r^2 \\
g_{12} &= g_{21} = 0
\end{aligned}
\tag{272}
$$

$$
(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{273}
$$

The determinant of (273) is $g = r^2$ hence $\sqrt{g} = r$. Note that because of (247) we have:

$$
(g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} \tag{274}
$$

We now plug the relevant bits into (241):

$$
\begin{aligned}
\nabla^2\Phi &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\Big(\sqrt{g}\, g^{kr}\frac{\partial\Phi}{\partial x^r}\Big) \\
&= \frac{1}{r}\Big[\frac{\partial}{\partial r}\Big(r.1\frac{\partial\Phi}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(r.\frac{1}{r^2}\,\frac{\partial\Phi}{\partial\theta}\Big)\Big] \\
&= \frac{1}{r}\,\frac{\partial}{\partial r}\Big(r\frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2}\,\frac{\partial^2\Phi}{\partial\theta^2} \\
&= \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\Phi}{\partial\theta^2}
\end{aligned}
\tag{275}
$$

which is the same as (45).

# 10 Appendix

## 10.1 Change of variables

If we suppose a change of variables as follows:
$\xi = \mathbf{B^t x}$ ie $\xi_r = \sum_{i=1}^n b_{ir} x_i$ for $r = 1, 2, \ldots, n$ we then have the following:

$$
\begin{aligned}
u_{x_i} &= \frac{\partial u}{\partial x_i} = \sum_{r=1}^{n} \frac{\partial u}{\partial \xi_r} \frac{\partial \xi_r}{\partial x_i} \\
&= \sum_{r=1}^{n} b_{ir} \frac{\partial u}{\partial \xi_r} \\
u_{x_i x_j} &= \frac{\partial}{\partial x_j} \Big( \frac{\partial u}{\partial x_i} \Big) \\
&= \sum_{s=1}^{n} (u_{x_i})_{\xi_s} \frac{\partial \xi_s}{\partial x_j} \\
&= \sum_{s=1}^{n} b_{js} (u_{x_i})_{\xi_s} \\
&= \sum_{s=1}^{n} b_{js} \frac{\partial}{\partial \xi_s} \Big( \sum_{r=1}^{n} b_{ir} \frac{\partial u}{\partial \xi_r} \Big) \\
&= \sum_{r,s=1}^{n} b_{ir} b_{js} \frac{\partial^2 u}{\partial \xi_r \partial \xi_s}
\end{aligned}
\tag{276}
$$

Equation (6) was as follows and is transformed in terms of the $\xi_i$ as follows:

$$
\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i,j=1}^{n} b_i u_{x_i} + cu = d \tag{277}
$$

So we have:

$$
\sum_{r,s=1}^{n} \underbrace{\left( \sum_{i,j=1}^{n} b_{ir} a_{ij} b_{js} \right)}_{c_{rs}} u_{\xi_r \xi_s} + \sum_{r=1}^{n} \left( \sum_{i=1}^{n} b_{ir} b_i \right) u_{\xi_r} + cu = d \tag{278}
$$

## 10.2 Proof of equation (227)

We have to prove:

$$
\nabla \,.\, (\Phi \mathbf{B}) = (\nabla \Phi) \,.\, \mathbf{B} + \Phi \,(\nabla \,.\, \mathbf{B}) \tag{279}
$$

We have from (194):

$$\nabla = \frac{\mathbf{e_1}}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{e_2}}{h_2}\frac{\partial}{\partial u_2} + \frac{\mathbf{e_3}}{h_3}\frac{\partial}{\partial u_3} \tag{280}$$

Therefore:

$$\begin{aligned}\nabla \mathbin{.} (\Phi\mathbf{A}) &= \begin{pmatrix} \frac{1}{h_1}\frac{\partial}{\partial u_1} \\ \frac{1}{h_2}\frac{\partial}{\partial u_2} \\ \frac{1}{h_3}\frac{\partial}{\partial u_3} \end{pmatrix} \mathbin{.} \begin{pmatrix} \Phi A_1 \\ \Phi A_2 \\ \Phi A_3 \end{pmatrix} \\ &= \frac{\Phi}{h_1}\frac{\partial A_1}{\partial u_1} + \frac{A_1}{h_1}\frac{\partial \Phi}{\partial u_1} + \frac{\Phi}{h_2}\frac{\partial A_2}{\partial u_2} + \frac{A_2}{h_2}\frac{\partial \Phi}{\partial u_2} + \frac{\Phi}{h_3}\frac{\partial A_3}{\partial u_3} + \frac{A_3}{h_3}\frac{\partial \Phi}{\partial u_3} \\ &= (\nabla\Phi) \mathbin{.} \mathbf{A} + \Phi(\nabla \mathbin{.} \mathbf{A})\end{aligned} \tag{281}$$

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# 12 History

Created 22/07/2018