The basics of Cesàro summability

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1 Introduction

Students of analysis are introduced to convergent series in a way which suggests that divergent series are so pathological that they are of no use and should be avoided at all costs. The very definition of what constitutes a convergent series arguably narrows one's view of potential alternative ways of looking at convergence. That definition is as follows. The series $\sum_{n=0}^{\infty} a_n$ is said to converge to the sum s if the partial sum $s_n = \sum_{n=0}^{n} a_n$ tends to a finite limit s when $n \to \infty$. A series which is not convergent is divergent.

We know that:

$$1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1 \tag{1}$$

Following Hardy ([1], page 2) we can seek to interpret (1) in a more general sense, not limited to the interval of x for which convergence is assured. Thus if s is the sum of the infinite series interpreted in this formal sense we should still have:

$$s = 1 + x + x^2 + x^3 + \cdots = 1 + x(1 + x + x^2 + \dots) = 1 + xs \quad \implies s = \frac{1}{1-x} \tag{2}$$

Note that the line of argument used in (2) does not involve any conditions on x and so there is some sense in which (1) could be said to be true for all x, leaving $x = 1$ as a special case, of course. If one "goes with the flow" we can put $x = e^{i\theta}$ and require that $0 < \theta < 2\pi$ so that $x \neq 1$. If we do this we get:

$$1 + e^{i\theta} + e^{2i\theta} + \cdots = \frac{1}{1 - e^{i\theta}} = \frac{1}{2} + i\cot(\frac{\theta}{2}) \tag{3}$$

Note that the final equality in (3) is derived by the following standard trigonometrical trick:

$$\begin{aligned}\frac{1}{1-e^{i\theta}} = \frac{1}{e^{\frac{i\theta}{2}}\,e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}\,e^{\frac{i\theta}{2}}} &= \frac{1}{e^{\frac{i\theta}{2}}\left(e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}\right)} \\ &= \frac{1}{e^{\frac{i\theta}{2}} \times -2i\sin\frac{\theta}{2}} \\ &= \frac{ie^{\frac{-i\theta}{2}}}{2\sin\frac{\theta}{2}} \\ &= \frac{i\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} \\ &= \frac{1}{2} + \frac{i}{2}\cot(\frac{\theta}{2})\end{aligned} \tag{4}$$

From (3) we can equate real and imaginary parts and we get (after expanding $e^{i\theta}, e^{2i\theta}$ etc) :

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots = 0 \tag{5}$$

$$\sin\theta + \sin 2\theta + \cdots = \frac{1}{2}\cot(\frac{\theta}{2}) \tag{6}$$

If we change θ to $\theta + \pi$ (5) and (6) become respectively:

$$\frac{1}{2} - \cos\theta + \cos 2\theta + \cdots = 0 \tag{7}$$

$$-\sin\theta + \sin 2\theta - \cdots = \frac{1}{2}\cot(\frac{\theta+\pi}{2}) = \frac{1}{2}\frac{\cos(\frac{\theta}{2}+\frac{\pi}{2})}{\sin(\frac{\theta}{2}+\frac{\pi}{2})} = \frac{-\sin(\frac{\theta}{2})}{2\cos(\frac{\theta}{2})} = -\frac{1}{2}\tan(\frac{\theta}{2}) \tag{8}$$

Hence:

$$\sin\theta - \sin 2\theta + \cdots = \frac{1}{2}\tan(\frac{\theta}{2}) \tag{9}$$

If we put $\theta = 0$ (which is equivalent to the problem case of $x = -1$ since we let $\theta \to \theta + \pi$ ie $e^{i\pi} = -1$) into (7) we get:

$$1 - 1 + 1 - \cdots = \frac{1}{2} \quad (10)$$

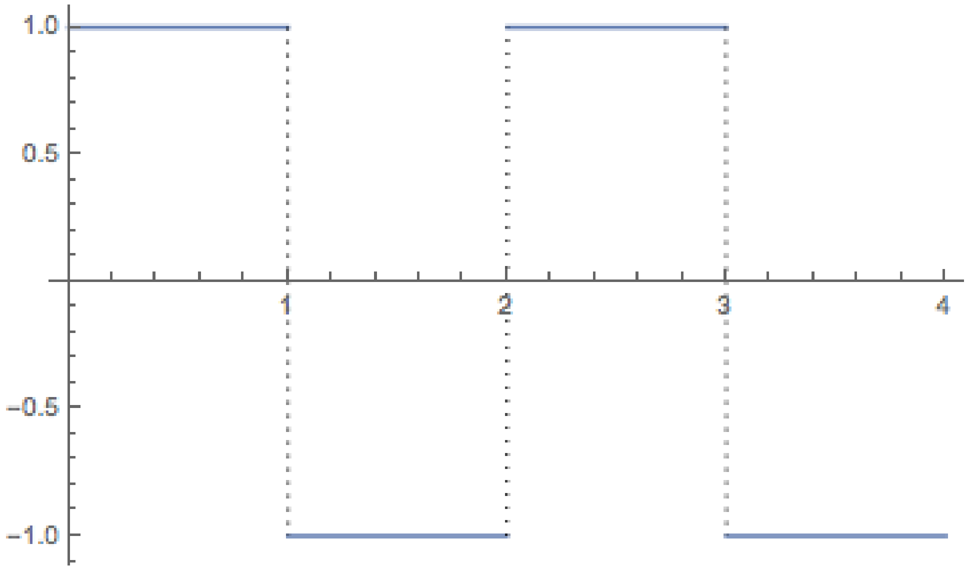
We can differentiate (7) and (9) repeatedly with respect to θ for θ such that $0 < \theta < \pi$ to get infinite series that look like this, for instance:

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{2k} \cos n\theta = 0 \quad k = 1, 2, \ldots \quad (11)$$

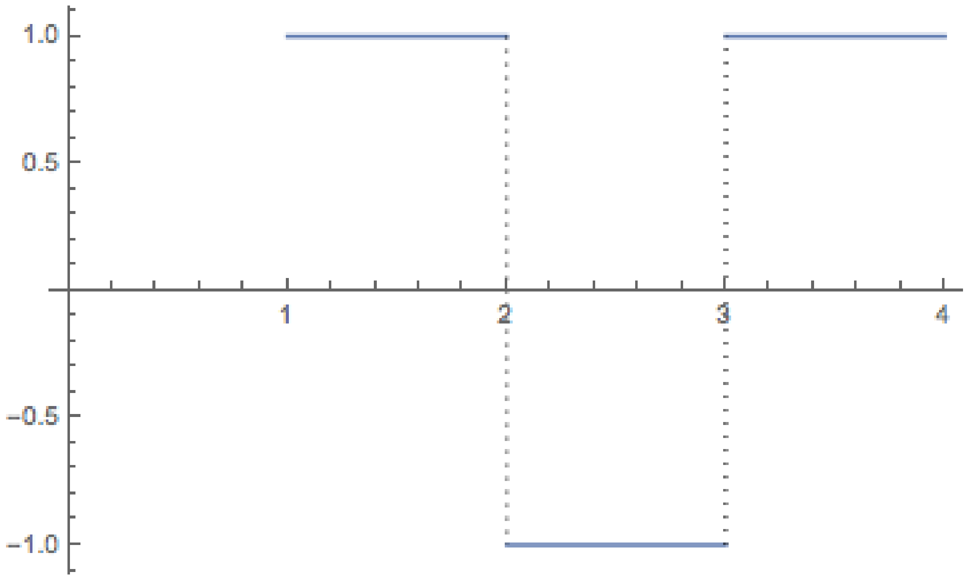
Hardy ([1], pages 2-5) lists a series of expressions like those above which are formal in nature but "are correct wherever they can be checked" ([1], page 5). What is going on here is a fundamentally different way of looking at what a "sum" of a series is. The modern approach is that mathematical symbolism has no inherent "meaning" - the meaning is given by definition. In the 18^{th} century the likes of Euler performed vast numbers of formal calculations like those given above (and much more complex ones that are set out in [1]) without ever really starting from a purely definitional basis. Hardy puts it this way: subject to some qualifications "it is broadly true to say that mathematicians before Cauchy asked not 'How shall we define 1-1+1- ...?', but "What is 1-1+1- ...?', and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal".

A physicist will undoubtedly ask what is the physical significance of a series such as $1 - 1 + 1 - \cdots = \frac{1}{2}$ and will usually be disappointed by the mathematician's answer. Indeed, Hardy spends some time commenting on Oliver Heaviside's long chapter on divergent series which was contained in the second volume of his *Electromagnetic Theory* published in 1899. Hardy says that Heaviside "is plainly not aware that, at the time when his volume was published, a scientific theory of divergent series already existed." ([1], page 36). Borel's work on divergent series dates from 1895-9 and Poincar*é's* theory of asymptotic series dates from 1886.

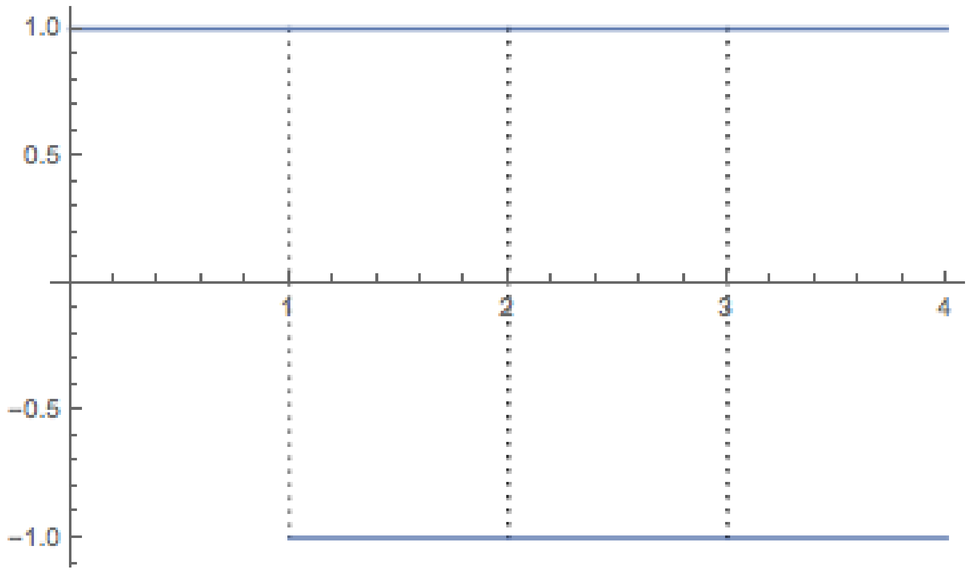
One way of trying to make physical sense of why a purely mathematical line of reason gives rise to $1 - 1 + 1 - \cdots = \frac{1}{2}$ is to consider a square wave signal that oscillates between $+1$ volts and -1 volts, say, at times $t = 0, 1, 2, \ldots$. Let's call this signal s and it looks like this:



Now suppose we simply delay signal *s* by one time unit so that it looks like this:



Now we add the two signals and the combined result looks like this:



For the interval $[1, \infty)$ the signals cancel and the net signal is +1 volt (on $[0, 1]$) averaged over two otherwise identical signals. Thus the mean (still a sum) of the signal is $\frac{1}{2}$. If you imagine two "infinite" power supplies producing the signals with the second one starting one time unit after the first started then there is some real physical sense in which the mean sum of the two signals is $\frac{1}{2}$ since the net +1 signal is the result of two separate one volt sources. However, that is perhaps as far as one could go.

The modern approach by physicists to divergent series and asymptotic expansions is worlds away from the context that Hardy was commenting on in the early part of the 20^{th} century. However, it is fair to say that in both most undergraduate mathematics and physics courses the concept of divergence is explained badly so that most students have no idea that there is actually a very developed theory of divergent series and asymptotic expansions. It is a bit like Laurent Schwartz's work on distributions which gave a rigorous foundation to the use of "functions" such as the Dirac delta function and Heaviside's step-function. Physicists used such functions on a daily basis without ever really worrying too much about a rigorous justification because it was clear enough to them why something like the Dirac delta "function" (being a limit of a family of Gaussians) seemed to work.

Looking at (10) one could argue as follows:

$$s = 1 - 1 + 1 - \cdots = 1 - (1 - 1 + 1 - \ldots) = 1 - s \quad \text{so } s = \frac{1}{2} \tag{12}$$

There is no hint of any limiting style of thinking in this argument - it is almost as though it is a visual gag - just move the brackets around in the original infinite series. The square wave analysis given above is perhaps a more concrete way of looking at what is going on. A method of summation, if it is to be of any use at all, ought to be *regular* in the sense that it sums every convergent series to its ordinary sum. In other words such a method of summation does not disturb the mathematical universe as we know it. Hardy ([1], page 6) postulates three axioms such a method should satisfy:

[A] If $\sum_n a_n = s$ then $\sum_n ka_n = ks$;

[B] If $\sum_n a_n = s$ and $\sum_n b_n = t$ then $\sum_n (a_n + b_n) = s + t$;

[C] If $a_0 + a_1 + a_2 + \cdots = s$ then $a_1 + a_2 + a_3 + \cdots = s - a_0$ and conversely.

Hardy notes that the manipulations in (12) satisfy [A] and [C] of the axioms.

If we consider the series:

$$1 - 2 + 3 - 4 + \cdots = s \tag{13}$$

We might perform the following formal manipulations:

$$\begin{aligned} s =& 1 - 2 + 3 - 4 + 5 - \cdots = 1 + (-2 + 3 - 4 + 5) + \dots) = 1 - (2 - 3 + 4 - 5 + \dots) \\ =& 1 - \Big[(1+1) - (2+1) + (3+1) - (4+1) + \dots\Big] \\ =& 1 - (1 + 1 - 2 - 1 + 3 + 1 - 4 - 1 + \dots) \\ =& 1 - \underbrace{(1 - 1 + 1 - \dots)}_{= \frac{1}{2} \text{ from (12)}} - (1 - 2 + 3 - 4 + \dots) = 1 - \frac{1}{2} - s \quad \text{so } s = \frac{1}{4} \end{aligned} \tag{14}$$

To arrive at (14) all of [A], [B] and [C] have been used.

Hardy describes four definitions of summation that arose historically and for the purposes of this article I will only refer to two (which both have relevance to Fourier theory), with the emphasis being on the frst method.

1.1 First method - Cesàro summability

The Cesàro sum is based on an average of partial sums as follows:

If $s_n = a_0 + a_1 + \cdots + a_n$ and

$$\lim_{n\to\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1} = s \tag{15}$$

then s is called the Cesàro sum of $\sum_n a_n$ and the Cesàro limit of s_n. The importance of the Cesàro limit is that if this limit exists it will equal the usual limit whenever that usual limit exists and may exist even if the usual limit does not exist. This is such an important result it needs to be proved.

As usual, take $\epsilon > 0$ and note that because $s_n \to s$ we can find an $N(\epsilon)$ such that $|s_n - s| < \frac{\epsilon}{2}$ for $n \geq N(\epsilon)$. We write $N(\epsilon)$ to emphasise the dependence of N on ϵ. Now let $Q = \sum_{j=0}^{N(\epsilon)} |s_n - s|$. Then:

$$\begin{aligned} \Big| \frac{1}{n+1} \sum_{j=0}^{n} s_j - s \Big| &= \frac{1}{n+1} \Big| \sum_{j=0}^{n} (s_j - s) \Big| \leq \frac{1}{n+1} \sum_{j=0}^{n} \big| s_j - s \big| \\ &= \frac{1}{n+1} \Big(\sum_{j=0}^{N(\epsilon)} \big| s_j - s \big| + \sum_{j=N(\epsilon)+1}^{n} \big| s_j - s \big| \Big) < \frac{1}{n+1} \Big(Q + (n - N(\epsilon)) \frac{\epsilon}{2} \Big) \end{aligned} \tag{16}$$

Now it is certainly true that $n - N(\epsilon) \leq n + 1$ and if we choose $M(\epsilon) \geq N(\epsilon)$ such that $M(\epsilon) \geq \frac{2Q}{\epsilon}$ it follows that $Q \leq \frac{\epsilon M(\epsilon)}{2}$. Thus the last line of (16) is :

$$\frac{1}{n+1} \Big(Q + (n - N(\epsilon)) \frac{\epsilon}{2} \Big) \leq \frac{1}{n+1} \Big((n+1) \frac{\epsilon}{2} + (n+1) \frac{\epsilon}{2} \Big) = \epsilon \tag{17}$$

This establishes that $\frac{1}{n+1} \sum_{j=0}^{n} s_j \to s$ as $n \to \infty$. For a slightly different proof see [2].

In the context of Fourier theory it was Fejér who showed that although the partial sums $S_N(f, t) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}$ might fail to converge, the Cesàro sum might behave better: $C_n(f, t) = \frac{1}{n+1} \sum_{j=0}^{n} S_j(f, t)$.

1.2 Second method -Abel summability

The second method, which also has importance in the context of Fourier theory, is Abel summability. A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said to be Abel summable to s if for every $0 \leq r < 1$, the series:

$$A(r) = \sum_{k=0}^{\infty} c_k r^k \tag{18}$$

converges and $\lim_{x \to 1} A(r) = s$.

For instance, if $1 - 2 + 3 - 4 + 5 - \cdots = \sum_{k=0}^{\infty} (-1)^k (k+1)$ it can be shown that the series is Abel summable to $\frac{1}{4}$. This is demonstrated as follows.

$$\begin{aligned}
A(r) =& 1 - 2r + 3r^2 - 4r^3 + 5r^4 - 6r^5 + \dots \\
r\,A(r) =& r - 2r^2 + 3r^3 - 4r^4 + 5r^5 - \dots \\
\therefore (1+r)\,A(r) =& 1 - r + r^2 - r^3 + r^4 - r^5 + \dots \\
=& 1 - (r + r^3 + r^5 + \dots) + (r^2 + r^4 + r^6 + \dots) \\
=& 1 - r(1 + r^2 + r^4 + r^6 + \dots) + (r^2 + r^4 + r^6 + \dots)
\end{aligned} \tag{19}$$

Now $1 + r^2 + r^4 + r^6 + \dots = \frac{1}{1-r^2}$ so that $r(1 + r^2 + r^4 + r^6 + \dots) = \frac{r}{1-r^2}$. Also, $r^2 + r^4 + r^6 + \dots = r^2(1 + r^2 + r^4 + \dots) = \frac{r^2}{1-r^2}$. Hence the last line of (19) becomes:

$$(1+r)\,A(r) = 1 - \frac{r}{1-r^2} + \frac{r^2}{1-r^2} = \frac{1-r}{1-r^2} \therefore A(r) = \frac{1-r}{(1+r)(1-r^2)} = \frac{1}{(1+r)^2} \tag{20}$$

As $r \to 1, A(r) \to \frac{1}{4}$. However, $A(r)$ is not Cesàro summable. It is shown in a series of exercises in [5, page 62] that:

convergent $\implies$ Cesàro summable $\implies$ Abel summable

Detailed proofs of these implications can be found in [2, pages 29-41]

Thus if $\sum_n a_n x^n$ is convergent for $0 \leq x < 1$ (thus x can be real or complex as long as $|x| < 1$) and its sum is $f(x)$ and:

$$\lim_{x \to 1-0} f(x) = s \tag{21}$$

then s is the Abel (A) sum of $\sum_n a_n$.

Note that $x \to 1 - 0$ means that x approaches 1 from below.

As shown in [5, page 55] in the context of Fourier theory the Abel sum or mean is such that it can be wriiten as a convolution. Thus if a function is represented by $f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n\, e^{in\theta}$ then the Abel mean is written as:

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}\, a_n\, e^{in\theta} \tag{22}$$

(22) can be written in terms of the Poisson kernel as follows:

$$A_r(f)(\theta) = (f * P_r)(\theta) \tag{23}$$

where the Poisson kernel is given by:

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \tag{24}$$

2 Some practical applications of Cesàro means

Cambridge University Fourier theory expert Tom Körner provides a series of exercises on Cesàro means in his book "Exercises for Fourier Analysis" [3]. That book accompanies Körner's well known and extremely interesting textbook [4] which, idiosyncratically, does not contain a single exercise. The exercises are essentially pitched at a Cambridge Tripos level student, so a student who struggles to get his or her mind around the basics of Fourier theory will probably be shattered by the first problem in the book.

2.1 Problem 1.1- [4] page 1

(i) Let $s_0 = \frac{1}{2}$, $s_n = \frac{1}{2} + \sum_{j=1}^{n-1} \cos jx$ for $n \geq 1$. By writing: $s_n = \frac{1}{2} \sum_{j=-n}^{n} e^{ijx}$ and summing geometric series show that $\frac{1}{n+1} \sum_{j=0}^{n} s_j \to 0$ as $n \to \infty$ for all $x \not\equiv 0 \mod 2\pi$, and so

$$0 = \frac{1}{2} + \sum_{j=1}^{\infty} \cos jx \quad \text{in the Cesàro sense.} \tag{25}$$

(ii) Show similarly that if $x \not\equiv 0 \mod 2\pi$, then:

$$\cot(\frac{x}{2}) = 2 \sum_{j=1}^{\infty} \sin jx \quad \text{in the Cesàro sense.} \tag{26}$$

Solution to Problem 1.1

The first part of the solution involves a standard result for the manipulation of trigonometrical series. Let $\omega = e^{ix}$.

$$\begin{aligned}
\sum_{j=-n}^{n} e^{ijx} &= \sum_{j=0}^{n} e^{ijx} + \sum_{j=-n}^{-1} e^{ijx} = \sum_{j=0}^{n} \omega^j + \sum_{j=-n}^{-1} \omega^j = \frac{1-\omega^{n+1}}{1-\omega} + \frac{\omega^{-n}-1}{1-\omega} \\
&= \frac{\omega^{-n} - \omega^{n+1}}{1-\omega} = \frac{\omega^{-n}(1-\omega^{2n+1})}{1-\omega} = \frac{\omega^{-n}(\omega^{\frac{2n+1}{2}}\,\omega^{\frac{-(2n+1)}{2}} - \omega^{\frac{2n+1}{2}}\,\omega^{\frac{2n+1}{2}})}{\omega^{\frac{1}{2}}\,\omega^{\frac{-1}{2}} - \omega^{\frac{1}{2}}\,\omega^{\frac{1}{2}}} \\
&= \frac{\omega^{-n}\,\omega^{\frac{2n+1}{2}}(\omega^{\frac{-(2n+1)}{2}} - \omega^{\frac{2n+1}{2}})}{\omega^{\frac{1}{2}}(\omega^{\frac{-1}{2}} - \omega^{\frac{1}{2}})} = \frac{-2i\sin\left((n+\frac{1}{2})x\right)}{-2i\sin(\frac{x}{2})} = \frac{\sin\left((n+\frac{1}{2})x\right)}{\sin(\frac{x}{2})}
\end{aligned} \tag{27}$$

Hence:

$$s_n = \frac{1}{2}\sum_{j=-n}^{n} e^{ijx} = \frac{\sin\left((n+\frac{1}{2})x\right)}{2\sin(\frac{x}{2})} = \frac{1}{2} + \sum_{j=1}^{n}\cos jx \tag{28}$$

Noting that $\frac{e^{ijx}+e^{-ijx}}{2} = \frac{2\cos jx}{2} = \cos jx$.

Now let:

$$K_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} s_j = \frac{1}{n+1}\sum_{j=0}^{n}\frac{\sin(j+\frac{1}{2})x}{2\sin(\frac{x}{2})} \tag{29}$$

We have to show that the RHS of (29) $\to 0$ as $n \to \infty$.

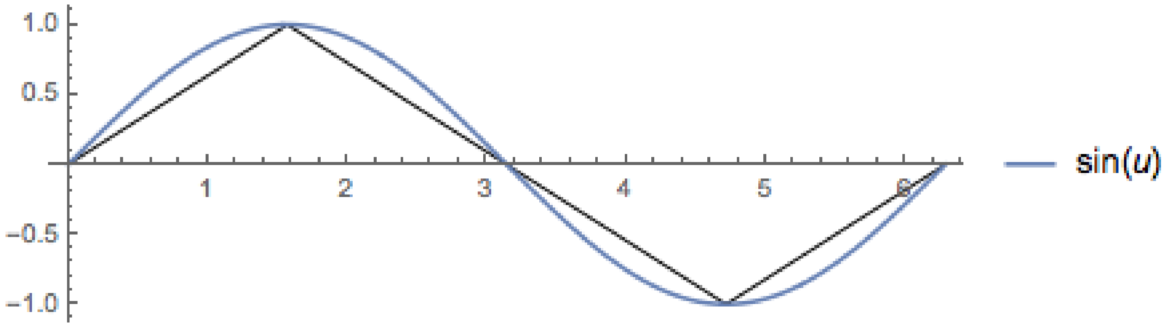
$$\begin{aligned}
K_n(x) = \frac{1}{n+1}\sum_{j=0}^{n}\frac{\sin(j+\frac{1}{2})x}{2\sin(\frac{x}{2})} \times \frac{2\sin(\frac{x}{2})}{2\sin(\frac{x}{2})} &= \frac{1}{n+1}\sum_{j=0}^{n}\frac{\sin jx\ \cos(\frac{x}{2})\,2\sin(\frac{x}{2}) + \cos jx\ \sin(\frac{x}{2})\,2\sin(\frac{x}{2})}{4\sin^2(\frac{x}{2})} \\
&= \frac{1}{4(n+1)\sin^2(\frac{x}{2})}\sum_{j=0}^{n}(\underbrace{\sin jx\ \sin x}_{\cos jx\ \cos x - \cos(j+1)x} + \underbrace{2\sin^2(\frac{x}{2})}_{1-\cos x}\cos jx) \\
&= \frac{1}{4(n+1)\sin^2(\frac{x}{2})}\sum_{j=0}^{n}[\cos jx\ \cos x - \cos(j+1)x + (1-\cos x)\ \cos jx] \\
&= \frac{1}{4(n+1)\sin^2(\frac{x}{2})}\sum_{j=0}^{n}[\cos jx - \cos(j+1)x] \\
&= \frac{1}{4(n+1)\sin^2(\frac{x}{2})}\Big[1 - \cos x + \cos x - \cdots + \cos(n-1)x - \cos(n+1)x\Big] \\
&= \frac{1-\cos(n+1)x}{4(n+1)\sin^2(\frac{x}{2})} \\
&= \frac{2\sin^2(\frac{n+1}{2})x}{4(n+1)\sin^2(\frac{x}{2})} \\
&= \frac{2}{n+1}\left(\frac{\sin(\frac{n+1}{2})x}{2\sin(\frac{x}{2})}\right)^2
\end{aligned} \tag{30}$$

Now for all $x \not\equiv 0 \mod 2\pi$ we have:

$$\frac{2}{n+1}\left(\frac{\sin(\frac{n+1}{2})x}{2\sin(\frac{x}{2})}\right)^2 \leq \frac{1}{2(n+1)}\frac{1}{\sin^2(\frac{x}{2})} \leq \frac{1}{2(n+1)}\frac{\pi^2}{l(x)^2} \to 0 \quad \text{as } n\to\infty \text{ for fixed } x \quad (31)$$

Here $l(x)$ is a linear function of x as explained below.

To understand the estimate in (31) consider the graph below which visually makes it clear that $|\sin u|$ is greater than the absolute value of the vertical displacement of the chords. For instance, for $u \in [0, \frac{\pi}{2}]$, $\sin u \geq \frac{2u}{\pi}$. For $u \in (\frac{1}{2}, \pi]$, $|\sin u| \geq \frac{2}{\pi}|u - \pi|$. More generally, for $\frac{k\pi}{2} < u < \frac{(k+1)\pi}{2}$, $|\sin u| \geq \frac{2}{\pi}|u - (k+1)\pi|$ for $k = 0, 1, 2, \ldots$.



Generally then, a relationship of the form $\frac{1}{2(n+1)}\frac{\pi^2}{l(x)^2}$ where $l(x) = \frac{x}{2} - (k+1)\pi$ exists for $k = 0, 1, 2, \ldots$ and for fixed x this goes to zero as $n \to \infty$.

What (31) shows is that $0 = \frac{1}{2} + \sum_{j=1}^{\infty} \cos jx$ in the Cesàro sense.

Now that we have done Part (i), part (ii) should hold no horrors of principle.

Solution to Part (ii)

Let $s_n = \sum_{j=1}^{n} \sin jx$ and $\omega = e^{ix}$ then:

$$\begin{aligned}
s_n &= \sum_{j=1}^{n} \frac{e^{ijx} - e^{-ijx}}{2i} = \frac{1}{2i}\sum_{j=1}^{n}(\omega^j - \omega^{-j}) = \frac{1}{2i(1-\omega)}\Big[\omega - \omega^{n+1} - (\omega^{-n} - 1)\Big] \\
&= \frac{1}{2i(1-\omega)}\Big[\omega(1-\omega^n) + 1 - \omega^{-n}\Big] = \frac{1}{2i\,(\omega^{\frac{1}{2}}\,\omega^{-\frac{1}{2}} - \omega^{\frac{1}{2}}\,\omega^{\frac{1}{2}})}\Big[\omega(\omega^{\frac{1}{2}}\,\omega^{-\frac{1}{2}} - \omega^{n-\frac{1}{2}}\,\omega^{\frac{1}{2}}) + \omega^{\frac{1}{2}}\,\omega^{-\frac{1}{2}} - \omega^{-n-\frac{1}{2}}\,\omega^{\frac{1}{2}}\Big] \\
&= \frac{1}{2i\,(\omega^{-\frac{1}{2}} - \omega^{\frac{1}{2}})}\Big[\omega(\omega^{-\frac{1}{2}} - \omega^{n-\frac{1}{2}}) + \omega^{-\frac{1}{2}} - \omega^{-n-\frac{1}{2}}\Big] = \frac{1}{2i}\Big[\frac{\omega^{\frac{1}{2}} + \omega^{-\frac{1}{2}} - (\omega^{n+\frac{1}{2}} + \omega^{-(n+\frac{1}{2})})}{-2i\sin(\frac{x}{2})}\Big] \\
&= \frac{2\cos(\frac{x}{2}) - 2\cos\left((n+\frac{1}{2})x\right)}{4\sin(\frac{x}{2})} = \frac{\cos(\frac{x}{2}) - \cos\left((n+\frac{1}{2})x\right)}{2\sin(\frac{x}{2})}
\end{aligned} \quad (32)$$

The Cesàro mean is given by (note here that $s_0 = 0$):

$$
\begin{aligned}
\frac{1}{n+1}\sum_{j=0}^{n} s_j =& \frac{1}{n+1}\underbrace{\sum_{j=0}^{n}\frac{\cos(\frac{x}{2})}{2\sin(\frac{x}{2})}}_{\text{note: n+1 terms}} - \frac{1}{n+1}\sum_{j=0}^{n}\frac{\cos\left((j+\frac{1}{2})x\right)}{2\sin(\frac{x}{2})} \\
=& \frac{\cot(\frac{x}{2})}{2} - \frac{1}{n+1}\sum_{j=0}^{n}\frac{\cos\left((j+\frac{1}{2})x\right)}{2\sin(\frac{x}{2})} \times \frac{2\sin(\frac{x}{2})}{2\sin(\frac{x}{2})} \\
=& \frac{\cot(\frac{x}{2})}{2} - \frac{1}{n+1}\sum_{j=0}^{n}\frac{\left(\cos jx\,\cos(\frac{x}{2})\,2\sin(\frac{x}{2}) - \sin jx\,\sin(\frac{x}{2})\,2\sin(\frac{x}{2})\right)}{4\sin^2(\frac{x}{2})} \\
=& \frac{\cot(\frac{x}{2})}{2} - \frac{1}{n+1}\sum_{j=0}^{n}\frac{\cos jx\,\sin x + \sin jx\,(\cos x - 1)}{4\sin^2(\frac{x}{2})} \\
=& \frac{\cot(\frac{x}{2})}{2} - \frac{1}{n+1}\sum_{j=0}^{n}\frac{(\cos jx\,\sin x + \sin jx\,\cos x - \sin jx)}{4\sin^2(\frac{x}{2})} \\
=& \frac{\cot(\frac{x}{2})}{2} - \frac{1}{n+1}\sum_{j=0}^{n}\frac{\left(\sin((j+1)x) - \sin jx\right)}{4\sin^2(\frac{x}{2})} \\
=& \frac{\cot(\frac{x}{2})}{2} - \frac{1}{4(n+1)\sin^2(\frac{x}{2})}\Big[(\sin x - \sin 0) + (\sin 2x - \sin x) + \cdots + (\sin nx - \sin((n+1)x)\Big] \\
=& \frac{\cot(\frac{x}{2})}{2} + \frac{\sin((n+1)x)}{4(n+1)\sin^2(\frac{x}{2})}
\end{aligned}
\tag{33}
$$

Now, for fixed x, as $n \to \infty$, $\frac{\sin((n+1)x)}{4(n+1)\sin^2(\frac{x}{2})} \to 0$ so that:

$$\frac{1}{n+1}\sum_{j=0}^{n} s_j \to \frac{\cot(\frac{x}{2})}{2} \quad \text{as } n \to \infty \tag{34}$$

But given the definition of $s_n = \sum_{j=1}^{n} \sin jx$ this means that $\cot(\frac{x}{2}) = 2\sum_{j=1}^{n} \sin jx$ in the Cesàro sense.

2.2 Problem 1.2, [4] page 1

(i) Suppose $s_j = (-1)^r(2r+1)$ for $r = 0, 1, 2, \ldots$. Show that $t_n = \frac{1}{n+1}\sum_{j=0}^{n} s_j$ does not tend to a limit but that $\frac{1}{n+1}\sum_{j=0}^{n} t_n$ does. In other words, applying the Cesàro procedure once does not produce a limit, but applying it twice does.

(ii) Give an example of a sequence where applying the Cesàro procedure twice does not produce a limit, but applying the Cesàro procedure three times does.

Solution to Problem 1.2 (i)

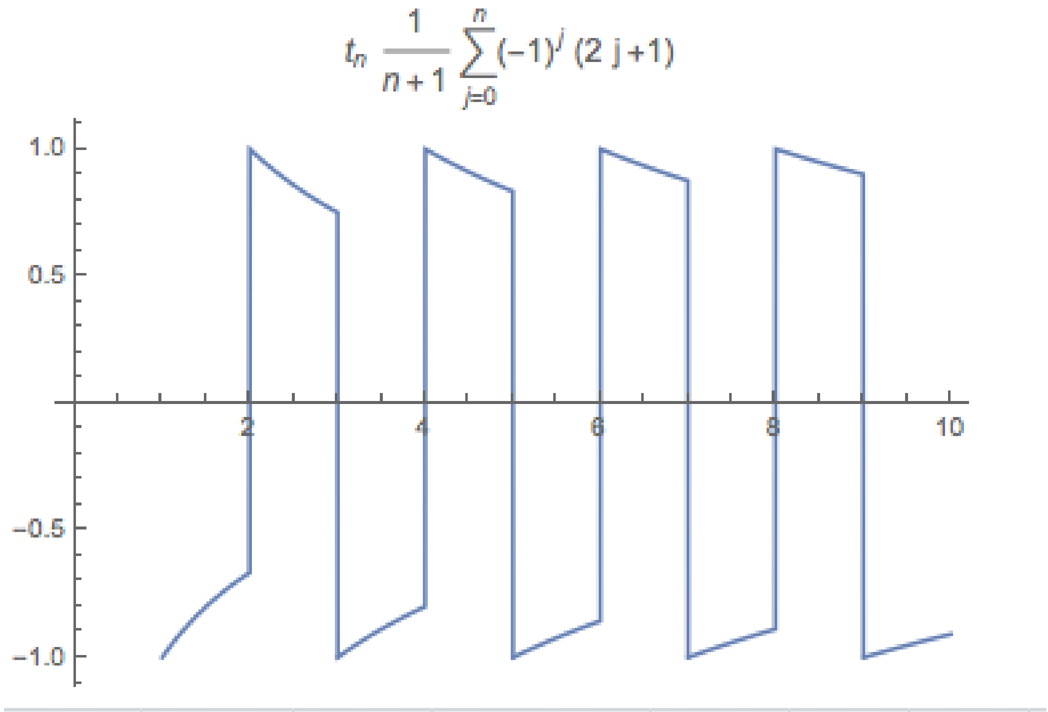
The mean $t_n = \frac{1}{n+1}\sum_{j=0}^{n} s_j$ oscillates between +1 and −1 depending on whether n is even or odd respectively. This can be seen as follows. Let $n = 2k$ then:

$$
\begin{aligned}
t_{2k} = \frac{1}{2k}\sum_{j=0}^{2k} s_j =& \frac{1}{2k+1}\Big[1-3+5-7+\cdots+(-1)^{2k-2}(4k-3)+(-1)^{2k-1}(4k-1)+(-1)^{2k}(4k+1)\Big] \\
=& \frac{1}{2k+1}\Big[1-3+5-7+\cdots+(-1)^{2k-2}[(4k-3)-(4k-1)]+(4k+1)\Big] \\
=& \frac{1}{2k+1}\Big[\underbrace{-2+-2+\cdots+-2}_{\text{k pairs}}+4k+1\Big] \\
=& \frac{1}{2k+1}\Big[-2k+4k+1\Big] = \frac{2k+1}{2k+1} = 1
\end{aligned}
\tag{35}
$$

Now we suppose $n = 2k+1$, then:

$$
\begin{aligned}
t_{2k+1} =& \frac{1}{2k+2}\sum_{j=0}^{2k+1} s_j \\
=& \frac{1}{2k+2}\Big[1-3+5-7+\cdots+(-1)^{2k-2}(4k-3)+(-1)^{2k-1}(4k-1)+(-1)^{2k}(4k+1)+(-1)^{2k+1}(4k+3)\Big] \\
=& \frac{1}{2k+2}\Big[1-3+5-7+\cdots+(-1)^{2k-2}\underbrace{[(4k-3)-(4k-1)]}_{-2}+(-1)^{2k}\underbrace{[4k+1-(4k+3)]}_{-2}\Big] \\
=& \frac{1}{2k+2}\Big[\underbrace{-2+-2+\cdots+-2+-2}_{\text{k+1 pairs}}\Big] \\
=& \frac{1}{2k+2}\Big[-2(k+1)\Big] = -1
\end{aligned}
\tag{36}
$$

This graph shows the behaviour:



To show that applying the Cesàro procedure twice gives a legitimate limit we have to examine the behaviour of:

$$\sigma_n = \frac{1}{n+1} \sum_{j=0}^{n} t_j = \frac{1}{n+1} \sum_{j=0}^{n} \frac{1}{j+1} \left(\sum_{k=0}^{j} (-1)^k (2k+1) \right) \tag{37}$$

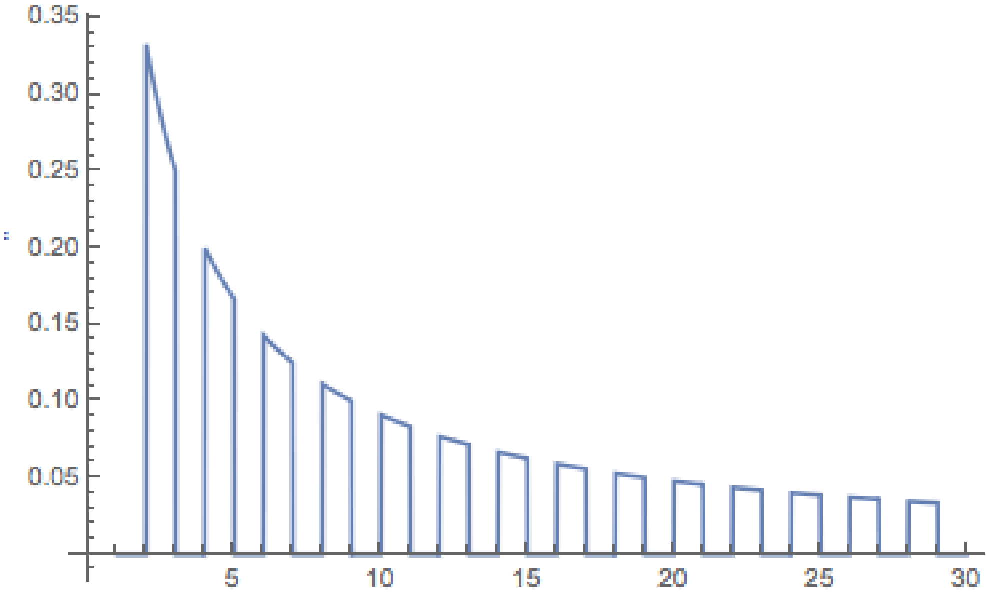
Using what we know from (35) and (36) we first suppose that n is even, say $n = 2r$. Then in the outer sum in (37) there are $2r + 1$ terms, $2r$ of which cancel because there are r "+1" terms and r " -1" terms. The remaining term is +1 so that $\sigma_n = \frac{1}{n+1}$.

Now if n is odd,say $n = 2r + 1$ there are $2r + 2$ terms in the outer sum of (37), giving rise to $(r + 1)$ terms of "+1" and $(r + 1)$ terms of "-1" which cancel out. Hence, $\sigma_n = 0$ in this case.

Thus we can say that $\sigma_n \to \frac{1}{n+1}$ as $n \to \infty$ because for any given $\epsilon > 0$ we can find an N such that $|\sigma_n - \frac{1}{n+1}| < \epsilon$ for all $n \geq N$.

Here is what σ_{30} looks like:

$$\sigma_{30} = \frac{1}{31} \sum_{j=0}^{30} \frac{1}{j+1} \left(\sum_{k=0}^{j} (-1)^k \ (2\,k+1) \right)$$



<u>Solution to Problem 1.2 (ii)</u>

An example of a sequence where two applications of the Cesàro procedure fails to produce a limit but a third application does is a sequence such as $s_r = (-1)^r(r^2 + 1)$ where the terms are quadratic in form.

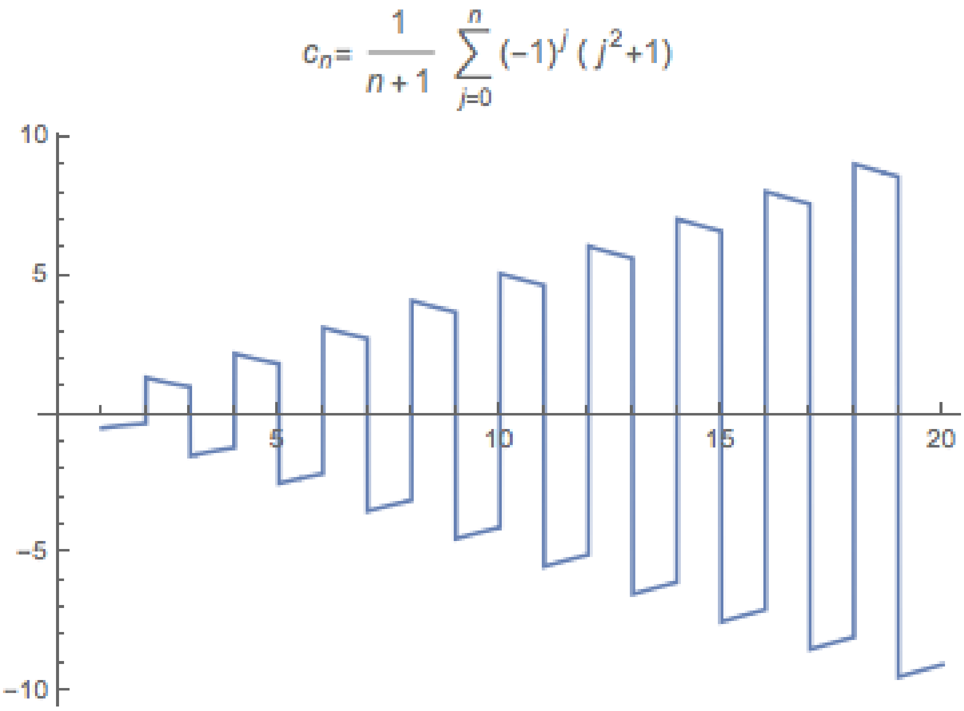
We can obtain closed form expressions for the outcome of each iteration of the Cesàro procedure as follows. We start with even $n = 2k$:

$$
\begin{aligned}
t_{2k} =& \frac{1}{2k+1}\sum_{j=0}^{2k} s_j = \frac{1}{2k+1}\sum_{j=0}^{2k}(-1)^j(j^2+1) = \frac{1}{2k+1}\Big\{ \underbrace{\sum_{j=0}^{2k}(-1)^j j^2}_{\sum_{j=0}^{2k} j^2 - 2\sum_{j=0}^{k-1}(2j+1)^2} + \underbrace{\sum_{j=0}^{2k}(-1)^j}_{=+1} \Big\} \\
=& \frac{1}{2k+1}\Big\{\sum_{j=0}^{2k} j^2 - 2\sum_{j=0}^{k-1}(2j+1)^2 + 1\Big\} = \frac{1}{2k+1}\Big\{\sum_{j=0}^{2k} j^2 - 2\sum_{j=0}^{k-1}(4j^2+4j+1)+1\Big\} \\
=& \frac{1}{2k+1}\Big\{\frac{(2k)(2k+1)(4k+1)}{6} - \frac{8(k-1)k(2k-1)}{6} - \frac{8(k-1)k}{2} - 2k + 1\Big\} \\
=& \frac{1}{2k+1}\Big\{\frac{k(2k+1)(4k+1) - 4(k-1)k(2k-1)}{3} - 4k^2 + 2k + 1\Big\} \qquad (38) \\
=& \frac{1}{2k+1}\Big\{k\Big[\frac{8k^2+6k+1-4(2k^2-3k+1)}{3}\Big] - 4k^2 + 2k + 1\Big\} \\
=& \frac{1}{2k+1}\Big\{k\Big[\frac{8k^2+6k+1-8k^2+12k-4}{3}\Big] - 4k^2 + 2k + 1\Big\} \\
=& \frac{1}{2k+1}\Big\{k\Big[\frac{18k-3}{3}\Big] - 4k^2 + 2k + 1\Big\} \\
=& \frac{1}{2k+1}\Big\{k(6k-1) - 4k^2 + 2k + 1\Big\} \\
=& \frac{2k^2+k+1}{2k+1}
\end{aligned}
$$

Clearly, as $k \to \infty$, $\frac{2k^2+k+1}{2k+1} \to \infty$ and hence we have divergence.

When n is odd ie $n = 2k+1$ we follow the same approach as above and we find that:

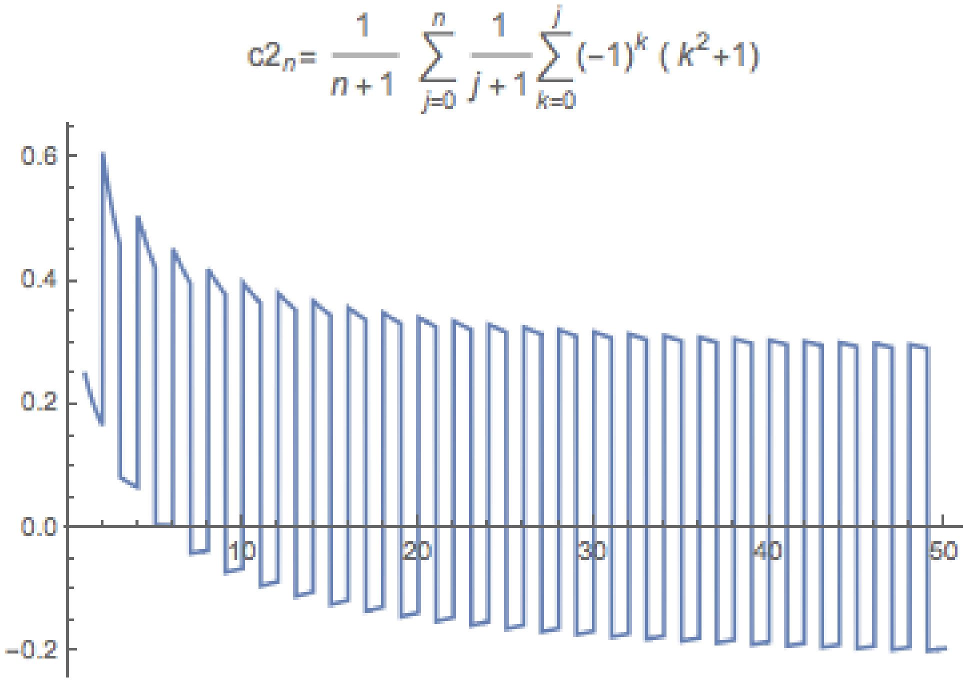
$$
t_{2k+1} = \frac{1}{2k+2}\sum_{j=0}^{2k+1}(-1)^j(j^2+1) = \frac{-(2k+1)}{2} \to \infty \text{ as } k \to \infty \qquad (39)
$$



When the Cesàro procedure is applied a second time the result is something which oscillates but the oscillation is reduced and the sum is damped. We can roughly estimate the effect of the second Cesàro procedure by multiplying (38) by $\frac{1}{2k+1}$ and (39) by $\frac{1}{2k+2}$ and observing what happens as $k \to \infty$:

$$\frac{2k^2+k+1}{(2k+1)^2} = \frac{2k^2+k+1}{4k^2+4k+1} \to \frac{1}{2} \quad \text{as } k \to \infty \tag{40}$$

$$\frac{-(2k+1)}{2(2k+2)} \to -\frac{1}{2} \quad \text{as } k \to \infty \tag{41}$$

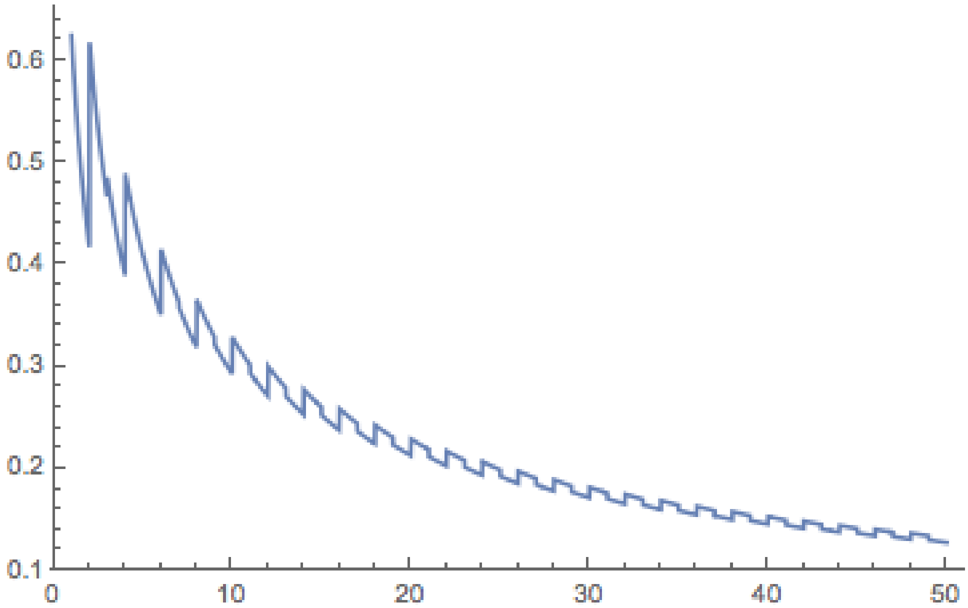


When the Cesàro procedure is applied a third time we get the further damping consistent with convergence. Thus;

$$\frac{2k^2+k+1}{(2k+1)^3} = \frac{2k^2+k+1}{8k^3+12k^2+6k+1} \to 0 \quad \text{as } k \to \infty \tag{42}$$

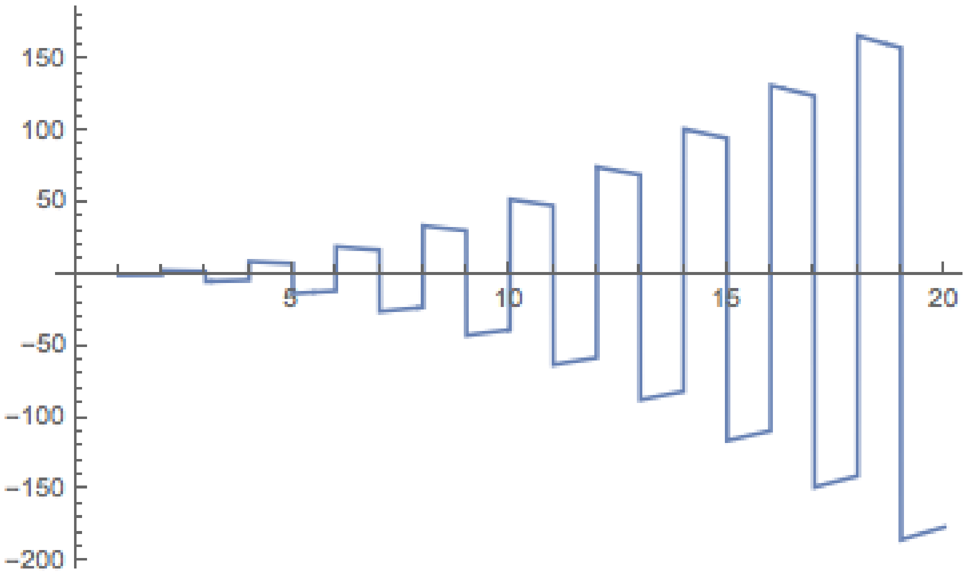
$$\frac{-(2k+1)}{2(2k+2)^2} \to 0 \quad \text{as } k \to \infty \tag{43}$$

$$c3_n = \frac{1}{n+1}\sum_{j=0}^{n}\frac{1}{j+1}\sum_{k=0}^{j}\frac{1}{k+1}\sum_{l=0}^{k}(-1)^l\,(l^2+1)$$

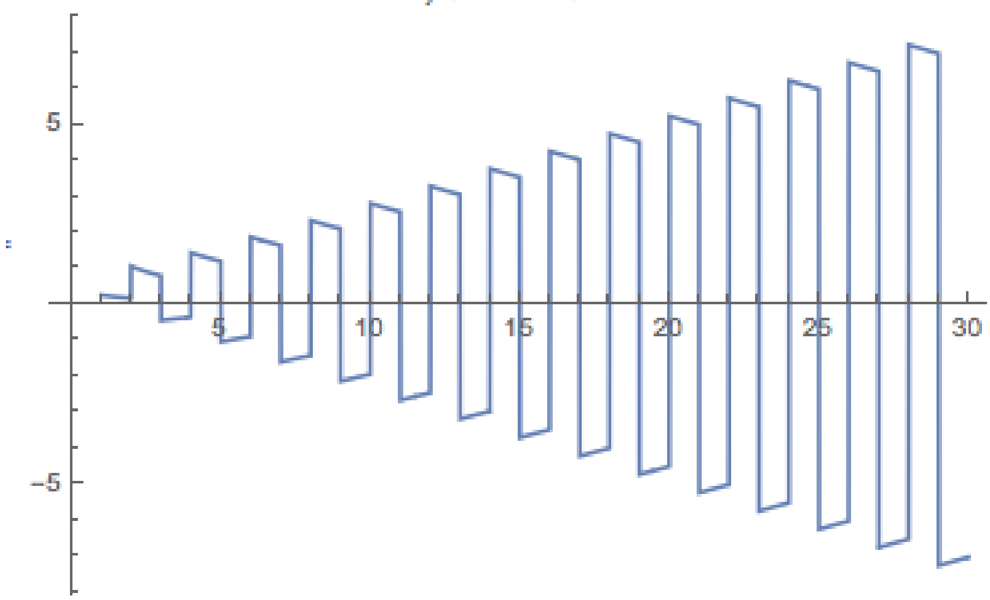


If we started with a cubic sequence $s_r = (-1)^r(r^3 + 1)$, say, we would have to apply the Cesàro procedure four times to get a limit:

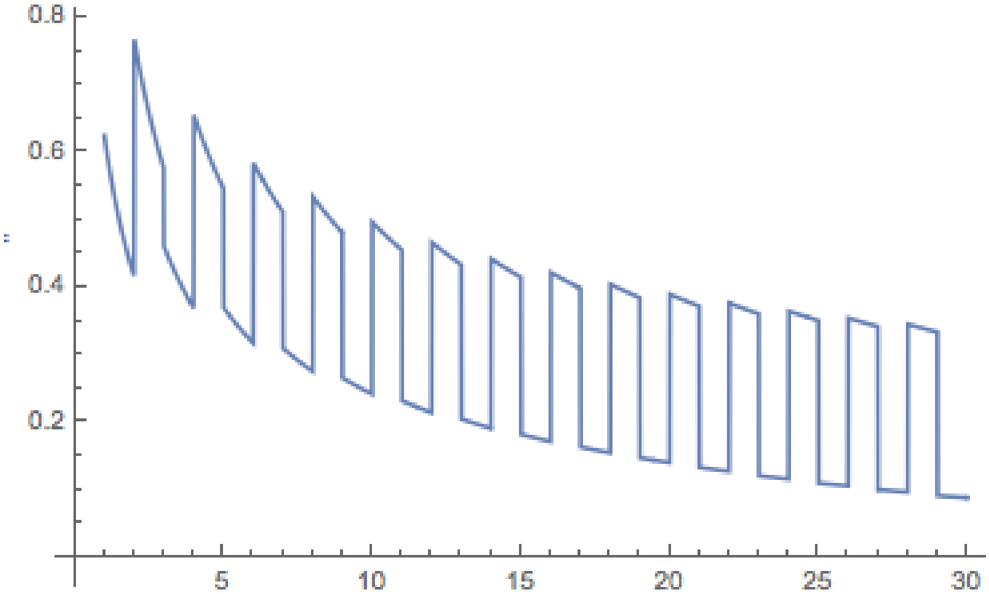
$$d1_n = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^j \, (j^3+1)$$



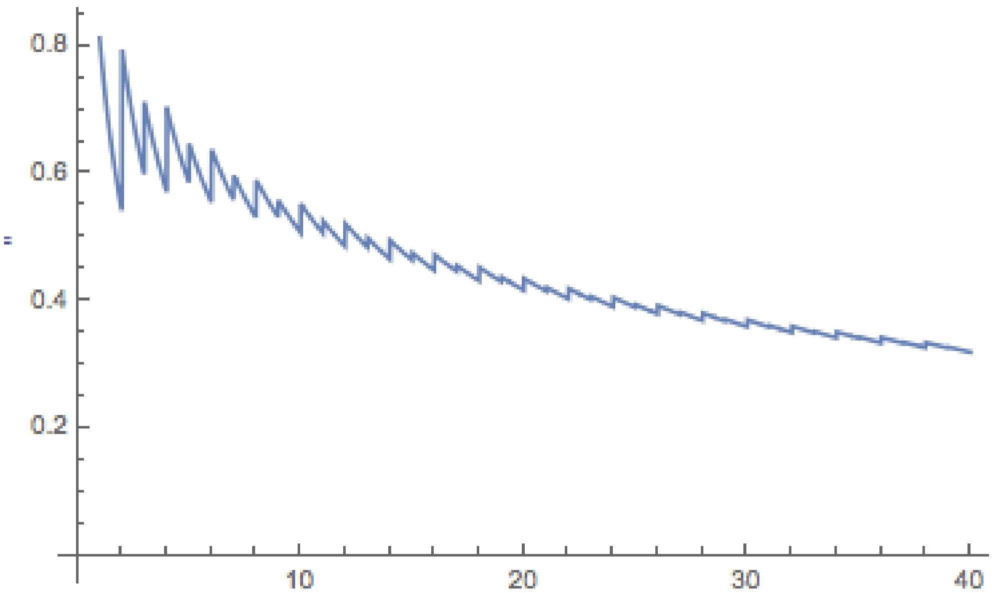
$$d2_n = \frac{1}{n+1} \sum_{j=0}^{n} \frac{1}{j+1} \sum_{k=0}^{j} (-1)^k \, (k^3+1)$$



$$d3_n = \frac{1}{n+1}\sum_{j=0}^{n}\frac{1}{j+1}\sum_{k=0}^{j}\frac{1}{k+1}\sum_{l=0}^{k}(-1)^l\,(l^3+1)$$



$$d4_n = \frac{1}{n+1}\sum_{m=0}^{n}\frac{1}{m+1}\sum_{j=0}^{m}\frac{1}{j+1}\sum_{k=0}^{j}\frac{1}{k+1}\sum_{l=0}^{k}(-1)^l\,(l^3+1)$$



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History:

04 March 2015: corrected typo