The binomial series for negative integral exponents

Peter Haggstrom
www.gotohaggstrom.com
mathsatbondibeach@gmail.com

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1 Background

Newton developed the binomial series in order to solve basic problems in calculus. Because the binomial series is such a fundamental mathematical tool it is useful to have a good grasp of it so that you can apply it in all the situations in which it arises. Problem 49 (`http://www.gotohaggstrom.com/Induction%20Problem%20Solution%20Set%2004%20May%202012.pdf`) gives a detailed proof of the convergence of the series and as it requires some relatively "hard core" real analysis it should probably bear a health warning! In this brief article all I want to deal with is the manipulation of the binomial series for negative integral exponents.

The binomial series for positive exponents gives rise to a finite number of terms ($n+1$ in fact if n is the exponent) and in its most general form is written as: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

2 The simple building block

We start with a simple "engine" for the development of negative exponents, namely, $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$. Note that the binomial factor is missing, That there is an infinity of terms can be established by simple long division ie:

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \qquad (1)$$

You may recognise (1) as a geometric series and you may also recall that it converges for $|x| < 1$. If you cannot remember how to prove this note that the n^{th} partial sum of

(1) is $s_n = 1 + x + x^2 + \cdots + x^{n-1}$. Thus $s_n - xs_n = 1 - x^n$ so that, finally, $s_n = \frac{1-x^n}{1-x}$. When $x = 1$ you get $s_n = n$ so that the series does not converge - the sum just gets bigger as n increases. When $x = -1$ you get an oscillatory series with $s_n = 0$ if n is even and $s_n = 1$ if n is odd.

When $-1 < x < 1$ we can write $|x| = \frac{1}{1+a}$ where $a > 0$. Then $|x|^n = \frac{1}{(1+a)^n}$.

Now since $a > 0$ we have by the binomial theorem:

$$(1+a)^n = 1 + \binom{n}{1}a + \binom{n}{2}a^2 + \cdots + \binom{n}{n}a^n > 1 + na \tag{2}$$

Thus $|x|^n = \frac{1}{(1+a)^n} < \frac{1}{1+na}$. Thus we can say that $x^n \to 0$ as $n \to \infty$ and this means that $s_n \to= \frac{1}{1-x}$ as $n \to \infty$.

3 Using the building block iteratively

To get $(1-x)^{-2}$, for instance, your instinct would be to square the convergent series for $(1-x)^{-1}$. This basic point seems pretty harmless but nevertheless requires some proof because there are some important principles at work. What is being asserted is that:

$$(1-x)^{-2} = (\sum_{k=0}^{\infty} x^k)(\sum_{k=0}^{\infty} x^k) = (1 + x + x^2 + x^3 + \dots)(1 + x + x^2 + x^3 + \dots) \tag{3}$$

To multiply two convergent infinite series you need a rule that guarantees the convergence of the product. There is a fundamental result known as Dirichlet's Theorem which states that the sum of a series of positive terms is the same no matter what order the terms are taken. The requirement that the terms are positive is not a problem because we know that (1) is convergent for $0 < |x| < 1$. Since the series$\sum_{k=0}^{\infty}|x|^k$ is convergent, the series $\sum_{k=0}^{\infty} x^k$ is convergent - this is what absolute convergence is all about.

To investigate absolute convergence a bit further consider a series of $\sum_{k=0}^{\infty} u_k$ where the u_k may be positive or negative. Thus we let $a_k = |u_k|$ which means that $a_k = u_k$ if u_k is positive and $a_k = -u_k$ if u_k is negative. Now let $v_k = u_k$ if $u_k > 0$ and $v_k = 0$ if $u_k < 0$. Also let $w_k = -u_k$ if $u_k < 0$ and $w_k = 0$ if $u_k > 0$. Then v_k and w_k are always positive. Now let $u_k = v_k - w_k$ and $a_k = v_k + w_k$. If $\sum_{k=0}^{\infty} a_k$ is convergent then both $\sum_{k=0}^{\infty} v_k$ and

$\sum_{k=0}^{\infty} w_k$ are convergent since they are merely selections from a convergent series of positive terms. Thus $\sum_{k=0}^{\infty} u_k = \sum_{k=0}^{\infty}(v_k - w_k)$ is convergent and equals $\sum_{k=0}^{\infty} v_k$ - $\sum_{k=0}^{\infty} w_k$.

The end result is that if a series is absolutely convergent, if you separate it into two series of positive and negative terms, these series are also convergent and the sum of the series is equal to the sum of the positive terms plus the sum of the negative terms.

4 Dirichlet's Theorem and its applications

Now let's apply Dirichlet's theorem. We start with a convergent series $\sum_{k=0}^{\infty} u_k$ of positive terms. We then rearrange this series in any order we like to form $\sum_{k=0}^{\infty} v_k$ so that the terms in this second series are identical to those in the first except for re-indexing. Let $\sum_{k=0}^{\infty} u_k = s$ so that the sum of any subset (including the entire infinity of terms) must be less than or equal to s. Now looking at $\sum_{k=0}^{\infty} v_k$, every term is also one of the u_k and hence when summed cannot exceed s. Thus $\sum_{k=0}^{\infty} v_k$ converges to some number $t \leq s$. A completely symmetrical argument shows that $s \leq t$ and putting the two results together it must be that $s = t$ ie the series converge to the same number.

When dealing with the product of two infinite series we can conceive of an infinite array of terms $u_m v_n$ which cover all possible product terms. Thus you get something like this:

$$\begin{array}{ccccc} u_0v_0 & u_1v_0 & u_2v_0 & u_3v_0 & \dots \\ u_0v_1 & u_1v_1 & u_2v_1 & u_3v_1 & \dots \\ u_0v_2 & u_1v_2 & u_2v_2 & u_3v_2 & \dots \\ u_0v_3 & u_1v_3 & u_2v_3 & u_3v_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

Dirichlet's Theorem can be used to prove the following:

If $u_0+u_1+u_2+\dots$ and $v_0+v_1+v_2+\dots$ are two convergent series of positive terms whose respective sums are s and t, then the series $u_0v_0+(u_1v_0+u_0v_1)+(u_2v_0+u_1v_1+u_0v_2)+\dots$ also converges and has sum st. To prove this we start with combinations of terms $u_m v_n$ such that $m+n=0$, $m+n=1$, $m+n=2$ etc. Thus if $m+n=0$ we have u_0v_0 while for $m+n=1$ we have u_1v_0 and u_0v_1. For $m+n=2$ we get u_2v_0, u_1v_1, u_0v_2. Adding these terms up with some suggestive grouping we get:

$$u_0v_0 + (u_1v_0 + u_0v_1) + (u_2v_0 + u_1v_1 + u_0v_2) + \dots \tag{4}$$

However, we can represent each of the terms in (4) as follows:

$$u_0v_0,\ (u_0+u_1)(v_0+v_1)-u_0v_0,\ (u_0+u_1+u_2)(v_0+v_1+v_2)-(u_0+u_1)(v_0+v_1)+\dots \quad (5)$$

When you add up the terms in (5) you get $(u_0+u_1+u_2)(v_0+v_1+v_2)$ and an inductive argument justfies the proposition that the sum of the first $n+1$ terms in (5) is:

$$(u_0+u_1+u_2+\cdots+u_n)(v_0+v_1+v_2+\cdots+v_n) \quad (6)$$

The infinite sum arising from (5) has limit st but it is nothing more than a rearrangement of the infinite series formed in (4). Dirichlet's Theorem then justifies the conclusion that (4) also convergs to st. The end result is that if we have two series $\sum_{k=0}^{\infty} u_k$ and $\sum_{k=0}^{\infty} v_k$ then:

$$\sum_{k=0}^{\infty} u_k \times \sum_{k=0}^{\infty} v_k = \sum_{k=0}^{\infty} w_k \quad \text{where } w_k = u_0v_k + u_1v_{k-1} + \cdots + u_kv_0 \quad (7)$$

So going back to (3) we have that:

$$\begin{aligned}(1-x)^{-2} = (\sum_{k=0}^{\infty} x^k)(\sum_{k=0}^{\infty} x^k) = (1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)\\ = 1+x+x^2+x^3+x^4+\dots\\ +x+x^2+x^3+x^4+\dots\\ +x^2+x^3+x^4+\dots\\ +x^3+x^4+\cdots+\dots\\ = 1+2x+3x^2+4x^3+\dots\end{aligned}$$

Thus because of the convergence of $\sum_{k=0}^{\infty} x^k$ we have that $(1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$ using the application of Dirichlet's Theorem immediately above.

Now do the same for $(1-x)^{-3}$, namely:

$$
\begin{aligned}
(1-x)^{-3} = (1-x)^{-2}(1-x)^{-1} = (1+2x+3x^2+4x^3+\dots)(1+x+x^2+x^3+\dots) \\
= 1+x+x^2+x^3+x^4+\dots \\
+2x+2x^2+2x^3+2x^4+\dots \\
+3x^2+3x^3+3x^4+\dots \\
+4x^3+4x^4+\dots \\
+5x^4+\dots \\
= 1+3x+6x^2+10x^3+15x^4+\dots
\end{aligned}
$$

The coefficient of x^n in $(1-x)^{-3}$ is found as follows:

$$1.x^n+2xx^{n-1}+3x^2x^{n-2}+\dots+nx^{n-1}x+(n+1)x^n = [1+2+3+\dots+(n+1)]x^n = \frac{(n+1)(n+2)}{2}x^n \tag{8}$$

5 Generalisation in a power series context

Using (7) w can generalise in a power series context as follows. Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$ when $|x| < R$ and $|x|$ is less than R or 1 then the following holds:

$$\frac{f(x)}{1-x} = \left(\sum_{k=0}^{\infty} a_k x^k\right)(1+x+x^2+\dots) = \sum_{k=0}^{\infty} s_k x^k \quad \text{where } s_k = a_0 + a_1 + \dots + a_k \tag{9}$$

Note here that (9) is in the same form as (7) where both of the series are convergent, either for $|x|$ less than R (in the case of f(x)) or 1 (in the case of $\frac{1}{1-x}$), because the coefficients of the series $1+x+x^2+\dots$ are all 1, we get the simplified form of s_k. Of course, x can be a complex number and all this logic is valid.

6 The formula and its proof

We are now in a position to develop the formula for negative integral indices. If n is a positive integer and $|x| < 1$ (and x may be complex) then:

$$\boxed{\frac{1}{(1-x)^n} = 1 + nx + \frac{n(n+1)}{1.2}\,x^2 + \cdots + \frac{n(n+1)\ldots(n+k-1)}{1.2\ldots k}\,x^k + \ldots} \tag{10}$$

For instance, checking this with n=3 we get:
$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \ldots$ as before.

To prove (10) assume it is true for all indicies up to n. We know from (9) with $f(x) = \frac{1}{(1-x)^n}$ that :

$$\frac{1}{(1-x)^{n+1}} = \left(\sum_{k=0}^{\infty} a_k x^k\right) \times (1 + x + x^2 + \cdots + x^k + \ldots) = \sum_{k=0}^{\infty} s_k x^k \tag{11}$$

where $s_k = a_0 + a_1 + \cdots + a_k = 1 + n + \frac{n(n+1)}{1.2} + \cdots + \frac{n(n+1)\ldots(n+k-1)}{1.2\ldots.k}$

Now a basic inductive argument shows that:

$$1 + n + \frac{n(n+1)}{1.2} + \cdots + \frac{n(n+1)\ldots(n+k-1)}{1.2.\ldots k} = \frac{(n+1)(n+2)\ldots(n+k)}{1.2\ldots k} \tag{12}$$

For $k = 1$ the LHS of (12) is n+1 while the RHS also equals $n+1$. Less trivially, when $k = 2$ the $LHS = 1 + n + \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2} = RHS$.
Assuming (12) as our inductive hypothesis we see that:

$$1 + n + \frac{n(n+1)}{1.2} + \cdots + \frac{n(n+1)\ldots(n+k-1)}{1.2.\ldots k} + \frac{n(n+1)\ldots(n+k-1)(n+k)}{1.2.\ldots k(k+1)} =$$
$$\frac{(n+1)(n+2)\ldots(n+k)}{1.2\ldots k}\left(1 + \frac{n}{k+1}\right) = \frac{(n+1)(n+2)\ldots(n+k)(n+k+1)}{1.2.\ldots k(k+1)} \tag{13}$$

Thus (12) holds for $k+1$ and (10) is a valid formula for the general form of the series expansion.