Tutorial on the uniform continuity of the Fourier Transform - Parts 1 and 2

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1 Introduction

An important property of the Fourier transform is its uniform continuity. How to prove this in a generalised setting is tricky since one has make suitable assumptions about the function space one is operating on. If we take something as commonplace in electrical engineering terms as a square wave or "box" function (ie something that is +1 on an interval such as $[-\frac{1}{2}, \frac{1}{2}]$ and zero elsewhere), when you just blindly do the Fourier transform you get something pretty amazing - a function which is continuous notwthstanding the discontinuities at $\pm\frac{1}{2}$ in the original function (see [2]). That remarkable function is sinc t $= \frac{\sin t}{t}$ and what is even more impressive is that when you take the inverse Fourier transform of sinc t you do get back to the original square wave, although to actually prove this rigorously isn't straighforward. It takes complex analysis involving contour integration and the residue theorem.

There are some basic points to make at this stage.

First it is a basic result that if $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$ then the Fourier transform and its inverse exist and are continuous. Thus the "box" function satisfies this criterion. However, because $\int_{-\infty}^{\infty} |\text{sinc t}|\, dt = \infty$ the hypothesis of the previous assertion is not satisfied, yet you can show that Fourier inversion does actually get you back to the original function. In other words you have to prove that $f(t) = \int_{-\infty}^{\infty} e^{2\pi i \xi t}\, \text{sinc}\, \xi\, d\xi$ with its jumps at $\pm\frac{1}{2}$. When you consider the basic elements of electrical engineering such as sine and cosine functions, the Fourier integrals of these do not even exist in the clasical sense - try $\int_{-\infty}^{\infty} e^{-2\pi i t \xi} \cos 2\pi t\, dt$ if you doubt me. It takes the theory of "tempered" distributions to ensure that Fourier transforms of all the "usual suspects" of electrical engineering actually make sense. This is where the Schwartz space of rapidly decreasing functions comes into its own. I am not going to go into how all of that works other than to say somewhat airily that it does, so you can take the Fourier transform of a cosine and get the Dirac delta "function" as an output and it is all kosher. Coupled with the Riemann-Lebesgue Lemma which ensures that the Fourier transform $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$, the uniform continuity of the Fourier transform holds no shocking surprises in behaviour for the transformed signal.

This tutorial is more in the classical vein of showing with epsilons and deltas etc that the Fourier transform is uniformly continuous. It is accompanied by videos:

Part 1: `https://www.youtube.com/watch?v=RG-dQMbGnMI&feature=youtu.be`

Part 2: `https://www.youtube.com/watch?v=iKWBYkPndZY&feature=youtu.be`

For more detail on Fourier integrals see [1].

2 PART 1

2.1 Theorem

If $f : \mathbb{R} \to \mathbb{C}$ is Riemann integrable on every interval $[a, b]$ and $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges, then $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t \xi}\, dt$ is uniformly continuous.

2.2 The recipe for the proof

First of all we actually need to show that the Fourier transform exists as a finite number. To do this we simply make the following calculation:

$$|\hat{f}(\xi)| = \Big| \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t \xi}\, dt \Big| \le \int_{-\infty}^{\infty} \Big| f(t)\, e^{-2\pi i t \xi} \Big|\, dt \le \int_{-\infty}^{\infty} |f(t)|\, dt < \infty \tag{1}$$

using the hypothesis of the theorem.

What we have to prove is uniform continuity which is a *global* rather than a local property in the sense that uniform continuity refers to a *set* of points rather than just one point. The logical statement of the property is as follows and it is useful to understand the order of the quantifiers because they are important:

$$\forall \epsilon > 0, \forall \xi, \eta \in \mathrm{X}, \exists \delta > 0 \text{ such that } |f(\xi) - f(\eta)| < \epsilon \text{ whenever } |\xi - \eta| < \delta \tag{2}$$

This is to be contrasted with ordinary continuity at a point x_0 say of a function $g(x)$. In that case the formulation is:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } |g(x) - g(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta \tag{3}$$

To illustrate the difference take a brutally simple case of $f(x) = x^2$ on $\mathbb{R}$. We know that f is continuous on $\mathbb{R}$ but is it *uniformly* continuous on $\mathbb{R}$?. If $f(x) = x^2$ were uniformly continuous we could take any two sufficiently close points as follows.

Let $x = n$ and $y = n + \frac{1}{n}$ where $n \in \mathbb{N}$. Hence for large n the two points can be made arbitrarily close. But $|f(y) - f(x)| = |(n + \frac{1}{n})^2 - n^2| = 2 + \frac{1}{n^2} > 2$. Because the definition has to work for *any* $\epsilon > 0$ this shows that the function cannot be uniformly continuous on $\mathbb{R}$ since no matter how close we make x and y, we can never get $|f(y) - f(x)| < 1$ for instance.

The main concept to take away is that with uniform continuity, you can give me any positive epsilon and I can find a delta for any two points in the set such that the absolute value of the difference in the Fourier transforms evaluated at those two points is less than epsilon.

In order to prove the theorem we have to get to a stage of showing that $|\hat{f}(\xi) - \hat{f}(\eta)| < \epsilon$ when $|\xi - \eta| < \delta$ which means that we have to somehow arrive at something that looks like this:

$$|\hat{f}(\xi) - \hat{f}(\eta)| < \text{constant } \times |\xi - \eta| \tag{4}$$

because if we make $|\xi - \eta| < \delta$ where $\delta \leq \frac{1}{\text{constant}}\epsilon$ then (4) guarantees that $|\hat{f}(\xi) - \hat{f}(\eta)| < \epsilon$.

The main idea is to break the domain of integration up into three parts which I call the "tails" and the "hump" and then use properties of the function for each of those pieces to make the absolute values of the contributions small enough. You need to use different properties for the hump and the tails as you will see. In what follows I define the Fourier transform as follows - and remember when using Mathematica, say, to check what parametrization you are using when comparing calculated results:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t}\, dt \tag{5}$$

We choose our $\epsilon > 0$ and pick a number $D(\epsilon)$ which we will use to break up the domain of integration. This number depends on ϵ because if we make ϵ very small we may have to make $D(\epsilon)$ very large to make the contributions of the tails sufficiently small, for instance.

$$\begin{aligned}
\Big|\hat{f}(\xi) - \hat{f}(\eta)\Big| &= \Big| \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t}\, dt - \int_{-\infty}^{\infty} f(t) e^{-2\pi i \eta t}\, dt \Big| \\
&= \Big| \int_{-\infty}^{\infty} f(t) \Big(e^{-2\pi i \xi t} - e^{-2\pi i \eta t} \Big)\, dt \Big| \\
&= \Big| \underbrace{\int_{|t| \geq D(\epsilon)} f(t) \Big(e^{-2\pi i \xi t} - e^{-2\pi i \eta t} \Big)\, dt}_{\text{tails}} + \underbrace{\int_{-D(\epsilon)}^{D(\epsilon)} f(t) \Big(e^{-2\pi i \xi t} - e^{-2\pi i \eta t} \Big)\, dt}_{hump} \Big| \\
&\leq \Big| \int_{|t| \geq D(\epsilon)} f(t) \Big(e^{-2\pi i \xi t} - e^{-2\pi i \eta t} \Big)\, dt \Big| + \Big| \int_{-D(\epsilon)}^{D(\epsilon)} f(t) \Big(e^{-2\pi i \xi t} - e^{-2\pi i \eta t} \Big)\, dt \Big| \\
&\leq \int_{|t| \geq D(\epsilon)} \Big| f(t) \Big(e^{-2\pi i \xi t} - e^{-2\pi i \eta t} \Big) \Big|\, dt + \int_{-D(\epsilon)}^{D(\epsilon)} \Big| f(t) \Big(e^{-2\pi i \xi t} - e^{-2\pi i \eta t} \Big) \Big|\, dt \\
&\leq \underbrace{\int_{|t| \geq D(\epsilon)} |f(t)|\, |e^{-2\pi i \xi t} - e^{-2\pi i \eta t}|\, dt}_{=\frac{\epsilon}{2}} + \int_{-D(\epsilon)}^{D(\epsilon)} |f(t)|\, |e^{-2\pi i \xi t} - e^{-2\pi i \eta t}|\, dt
\end{aligned} \tag{6}$$

From the hypotheses of the theorem we know that because $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges, it follows that for any $\epsilon > 0$ we can make the tails of the integral small - this is where the $D(\epsilon)$ comes in since its existence is guaranteed. Thus we can say that $\int_{|t| \geq D(\epsilon)} |f(t)|\, dt < \frac{\epsilon}{4}$. Now in the tails integral we can easily bound the other factor as follows $|e^{-2\pi i \xi t} - e^{-2\pi i \eta t}| \leq |e^{-2\pi i \xi t}| + |e^{-2\pi i \eta t}| \leq 1 + 1 = 2$. Thus the tails integral becomes:

$\int_{|t| \geq D(\epsilon)} |f(t)|\, |e^{-2\pi i \xi t} - e^{-2\pi i \eta t}|\, dt \leq 2 \times \frac{\epsilon}{4} = \frac{\epsilon}{2}$.

The hump is more tricky since we can't necessarily say that area in the hump is small particularly if we had to make $D(\epsilon)$ big in order to get the tails sufficiently small. Moreover, we have the other exponential component and recall that we had to somehow get a factor $|\xi - \eta|$ in the overall result. Now because f is integrable on any interval it is bounded on that interval and is thus bounded on $[-D(\epsilon), D(\epsilon)]$. Thus we can say that there is a $B(\epsilon) > 0$ such that:

$$\sup_{t \in [-D(\epsilon), D(\epsilon)]} = B(\epsilon)$$

The bound depends on ϵ because if we vary ϵ, we expect the bound to vary. We want to get an estimate for the hump which is useful. This involves massaging the exponential factor as follows. We start with a general property of exponentials (for any real a, b) which is the guts of the estimate:

$$|e^{ia} - e^{ib}| \leq |a - b| \tag{7}$$

This is proved as follows:

$$\begin{aligned}
|e^{ia} - e^{ib}| =& |\cos a + i \sin a - \cos b - i \sin b| \\
=& |\cos a - \cos b + i(\sin a - \sin b)| \\
=& \sqrt{(\cos a - \cos b)^2 + (\sin a - \sin b)^2} \\
=& \sqrt{\cos^2 a - 2\cos a\, \cos b + \cos^2 b + \sin^2 a - 2 \sin a\, \sin b + \sin^2 b} \\
=& \sqrt{2(1 - \cos(a - b))} \\
=& \sqrt{2 \times 2 \sin^2 \left(\frac{a-b}{2}\right)} \\
=& 2 \times \sin\left(\frac{a-b}{2}\right) \\
\leq& 2 \times \left|\sin\left(\frac{a-b}{2}\right)\right| \\
\leq& 2 \times \frac{|a-b|}{2} \\
=& |a - b|
\end{aligned} \tag{8}$$

We could also prove this result by noting that $\left|\int_b^a e^{i\theta}\, d\theta\right| = \left|\frac{e^{ia} - e^{ib}}{i}\right| = |e^{ia} - e^{ib}|$ but $\left|\int_b^a e^{i\theta}\, d\theta\right| \leq \int_b^a |e^{i\theta}|\, d\theta = \int_b^a d\theta = a - b \leq |a - b|$.

Note here that $|\sin x| \leq |x|$ for all x (draw a diagram to convince yourself). Thus we have the following:

$$\begin{aligned}
|e^{-2\pi i \xi t} - e^{-2\pi i \eta t}| \leq& |-2\pi\xi t - (-2\pi\eta t)| \\
=& 2\pi|t|\, |\xi - \eta|
\end{aligned} \tag{9}$$

(Note:$|x| = |-x|$)

So over $[-D(\epsilon), D(\epsilon)]$ we can dominate the integrand in the hump as follows:

$$\begin{aligned}
|f(t)|\, |e^{-2\pi i \xi t} - e^{-2\pi i \eta t}| \leq& \sup_{t \in [-D(\epsilon), D(\epsilon)]} \{2\pi|t|\, |\xi - \eta|\, |f(t)|\} \\
\leq& 2\pi \times 2D(\epsilon) \times B(\epsilon) \times |\xi - \eta|
\end{aligned} \tag{10}$$

So what we have now in the hump integral is this:

$$\begin{aligned}\int_{-D(\epsilon)}^{D(\epsilon)} |f(t)|\,|e^{-2\pi i\xi t} - e^{-2\pi i\eta t}|\,dt \leq &\int_{-D(\epsilon)}^{D(\epsilon)} 2\pi \times 2D(\epsilon) \times B(\epsilon) \times |\xi - \eta|\,dt \\ \leq &8\pi D^2(\epsilon)B(\epsilon)|\xi - \eta|\end{aligned} \tag{11}$$

So finally we have the following overall estimate (see last line of (6)):

$$\left|\hat{f}(\xi) - \hat{f}(\eta)\right| \leq \frac{\epsilon}{2} + 8\pi D^2(\epsilon)B(\epsilon)|\xi - \eta| \tag{12}$$

So if make $|\xi - \eta| \leq \frac{\epsilon}{16\pi D^2(\epsilon)B(\epsilon)+1}$ we finally have:

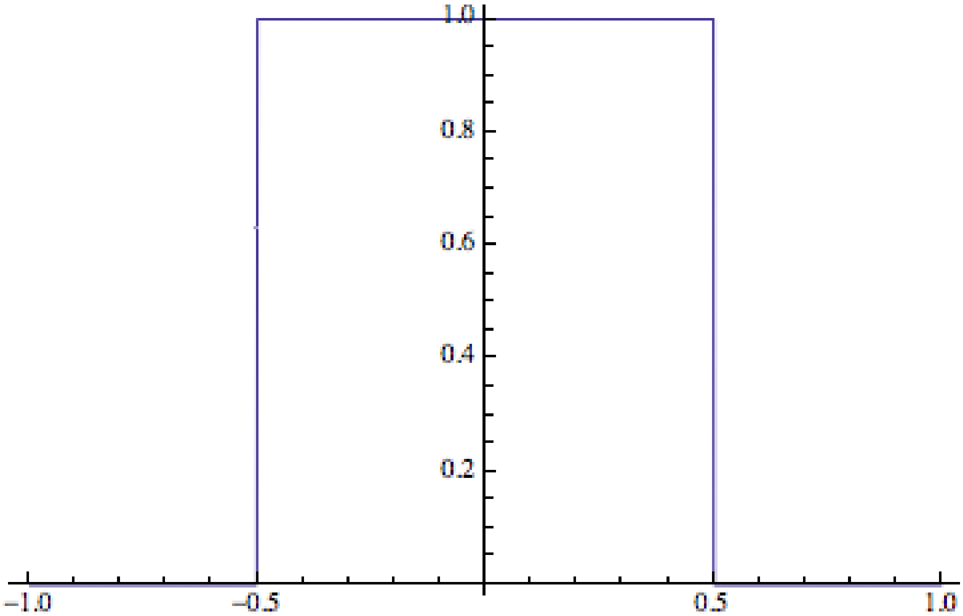
$$\begin{aligned}\left|\hat{f}(\xi) - \hat{f}(\eta)\right| \leq &\frac{\epsilon}{2} + 8\pi D^2(\epsilon)B(\epsilon)|\xi - \eta| \\ \leq &\frac{\epsilon}{2} + 8\pi D^2(\epsilon)B(\epsilon) \times \frac{\epsilon}{16\pi D^2(\epsilon)B(\epsilon)+1} \\ < &\epsilon\end{aligned} \tag{13}$$

This is what we wanted to show and if you were given a specific case you could in principle demonstrate all the estiimates in detail. Remember the Fourier transform of the box function was the sinc function so we now know that the sinc function is uniformly continuous on the real line (compare with the approach in [2]).

3 PART 2

Although the Fourier transfrom seems well behaved, at least by the standards of uniform continuity, it does necessarily behave well in general. For instance, let's return to our simple box function defined as follows:

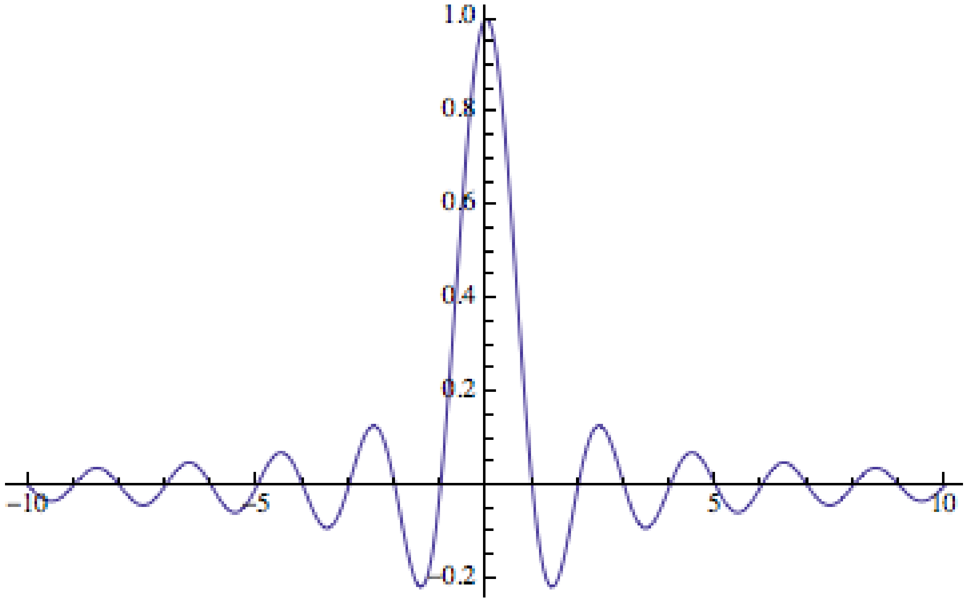
$$f(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2} \\ 0 & \text{if } |t| > \frac{1}{2} \end{cases} \tag{14}$$



Clearly f satisfies the assumptions of the theorem because it is Riemann intergrable on every interval and $\int_{-\infty}^{\infty} |f(t)|\, dt = 1$. Hence its Fourier transform is uniformly continuous. The Fourier transform is simply:

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}\, dt \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i\xi t}\, dt \\ &= -\frac{1}{2\pi i\xi}\left[e^{-\pi i\xi} - e^{\pi i\xi}\right] \\ &= \frac{\sin \pi\xi}{\pi\xi} \\ &= \text{sinc } \pi\xi\end{aligned} \tag{15}$$

The graph of sinc $\pi\xi$ looks like this:



We know that $\int_{-\infty}^{\infty} \frac{\sin \pi\xi}{\pi\xi}\, d\xi = 1$ (see [1] or any Fourier theory textbook) but does $\int_{-\infty}^{\infty} \left|\frac{\sin \pi\xi}{\pi\xi}\right| d\xi$ converge? If you follow why the integral of the sinc function converges (there is a lot of cancellation going on because of alternating signs) then you might bet that the integral of its absolute value does not converge. In essence the damping factor $\frac{1}{|\xi|}$ does not decay fast enough to guarantee convergence. The convergence of trigonometrical integrals is a very subtle matter and if you want to challenge yourself read Zygmund's tome ([3]). To establish the divergence we have to show the following:

$$\int_{-M}^{M} \left|\frac{\sin \pi\xi}{\pi\xi}\right| d\xi \to \infty \text{ as } M \to \infty \tag{16}$$

Because we are working with a trigonometrical function we choose $M = (N + 1)$ for $N \in \{1, 2, 3, \dots\}$. The reason for this choice will become clear shortly. We want to show that the integral gets arbitrarily big and so we need to show that it is at least as big as something that gets big. Thus we have the following:

$$
\begin{aligned}
\int_{-(N+1)}^{N+1} \Big|\frac{\sin \pi\xi}{\pi\xi}\Big|\, d\xi &\geq \sum_{k=0}^{N} \int_{k+\frac{1}{4}}^{k+\frac{3}{4}} \Big|\frac{\sin \pi\xi}{\pi\xi}\Big|\, d\xi \\
&= \sum_{k=0}^{N} \int_{k\pi+\frac{\pi}{4}}^{k\pi+\frac{3\pi}{4}} \Big|\frac{\sin \xi}{\pi\xi}\Big|\, d\xi \quad (\xi \to \pi\xi) \\
&\geq \sum_{k=0}^{N} \int_{k\pi+\frac{\pi}{4}}^{k\pi+\frac{3\pi}{4}} \frac{1}{\pi}\frac{1}{\sqrt{2}}\frac{1}{\xi}\, d\xi \\
&\geq \sum_{k=0}^{N} \int_{k\pi+\frac{\pi}{4}}^{k\pi+\frac{3\pi}{4}} \frac{1}{\pi}\frac{1}{\sqrt{2}}\frac{1}{(k\pi+\frac{3\pi}{4})}\, d\xi \\
&\geq \sum_{k=0}^{N} \int_{k\pi+\frac{\pi}{4}}^{k\pi+\frac{3\pi}{4}} \frac{1}{\pi^2}\frac{1}{\sqrt{2}}\frac{1}{(k+1)}\, d\xi \\
&\geq \sum_{k=0}^{N} \frac{1}{\pi^2}\frac{1}{\sqrt{2}}\frac{1}{(k+1)}\frac{\pi}{2} \\
&= \sum_{k=0}^{N} \frac{1}{\pi}\frac{1}{2\sqrt{2}}\frac{1}{(k+1)} \to \infty \text{ as } N \to \infty
\end{aligned} \tag{17}
$$

Note that on the intervals concerned eg $[\frac{\pi}{4}, \frac{3\pi}{4}]$, $\sin\xi \geq \frac{1}{\sqrt{2}}$ (just draw a diagram to convince yourself). Similarly when seeking to dominate $\frac{1}{\xi}$ we take the righthand endpoint of the interval since the function is lowest there. The final sum is a harmonic sum which is proved in first year analysis to diverge.

4 References

[1] Peter Haggstrom, *Basic Fourier Integrals*, `http://www.gotohaggstrom.com/Basic%20Fourier%20integrals.pdf`
[2] Peter Haggstrom, *Uniform continuity of sinc x* , `http://www.gotohaggstrom.com/Uniform%20continuity%20of%20sinc%20x.pdf`
[3] A Zygmund, *Trigonometric Series*, Third Edition, Vols 1 and 2 combined, Cambridge University Press, 2002)

5 History

Created 02/01/2017
27/09/2022 Added short integration proof for the inequality in (8)