# Vieta's formula

Peter Haggstrom
www.gotohaggstrom.com
mathsatbondibeach@gmail.com

September 6, 2018

## 1 Introduction

Vieta's formula is one of those visually pleasing mathematical relationships that also captures some important concepts which were systematically investigated by Mark Kac in his well known work on statistical independence [1]

Here is the formula:

$$\begin{aligned} \frac{2}{\pi} &= \prod_{n=1}^{\infty} \cos \frac{\pi}{2^{n+1}} \\ &= \frac{\sqrt{2}}{2} \, \frac{\sqrt{2+\sqrt{2}}}{2} \, \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \dots \end{aligned} \tag{1}$$

The RHS of (1) follows from the half angle formula as follows:

$$\frac{\sqrt{2}}{2} = \cos \frac{\pi}{4} = \cos \frac{2\pi}{8} = 2\cos^2 \frac{\pi}{8} - 1 \tag{2}$$

Therefore:

$$\begin{aligned} \cos \frac{\pi}{8} &= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} \\ &= \sqrt{\frac{2+\sqrt{2}}{4}} \\ &= \frac{\sqrt{2+\sqrt{2}}}{2} \end{aligned} \tag{3}$$

Also:

$$\cos\frac{\pi}{8} = \cos\frac{2\pi}{16} = 2\cos^2\frac{\pi}{16} - 1 \tag{4}$$

Therefore:

$$\begin{aligned}\cos\frac{\pi}{16} &= \sqrt{\frac{\cos\frac{\pi}{8}+1}{2}} \\ &= \sqrt{\frac{\frac{\sqrt{2+\sqrt{2}}}{2}+1}{2}} \\ &= \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\end{aligned} \tag{5}$$

and so on.

To derive (1) we start with the half angle formula for $\sin x$ and iterate:

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\ &= 2^2\sin\frac{x}{4}\cos\frac{x}{4}\cos\frac{x}{2} \\ &= 2^3\sin\frac{x}{8}\cos\frac{x}{8}\cos\frac{x}{4}\cos\frac{x}{2} \\ &\vdots \\ &= 2^n\sin\frac{x}{2^n}\prod_{k=1}^{\infty}\cos\frac{x}{2^k}\end{aligned} \tag{6}$$

If you doubt the last step in (6) please feel free to do a quick inductive proof.

We know that for $x \neq 0$:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \tag{7}$$

and this justisfies the following:

$$1 = \lim_{n\to\infty}\frac{\sin\frac{x}{2^n}}{\frac{x}{2^n}} = \frac{1}{x}\lim_{n\to\infty}2^n\sin\frac{x}{2^n} \tag{8}$$

and so

$$\lim_{n\to\infty}2^n\sin\frac{x}{2^n} = x \tag{9}$$

Now combining (6) and (9) we have:

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \cos \frac{x}{2^k} \tag{10}$$

and setting $x = \frac{\pi}{2}$ we get Vieta's formula (1).

In what follows I follow Kac's development of Vieta's formula in the context of statistical independence and, in so doing, flesh out the details he omits.

## 2 Binary expansion using Rademacher functions

Every real number $t$ such that $0 \le t \le 1$ can be written uniquely in the form:

$$t = \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \dots \tag{11}$$

where each $\epsilon_k$ is either 0 or 1. To ensure uniqueness the convention is to write terminating expansions in the form in which all digits from a certain point on are 0.

Because the $\epsilon_k$ are actually functions of $t$ we can explicitly denote this by writing (11) in the following form:

$$t = \frac{\epsilon_1(t)}{2} + \frac{\epsilon_2(t)}{2^2} + \dots \tag{12}$$

The $\epsilon_k(t)$ are step functions and $\epsilon_1(t)$, for instance, is set out in Figure 1 where $\epsilon(t) = 0$ for $0 \le t < \frac{1}{2}$ and 1 for $\frac{1}{2} \le t < 1$.

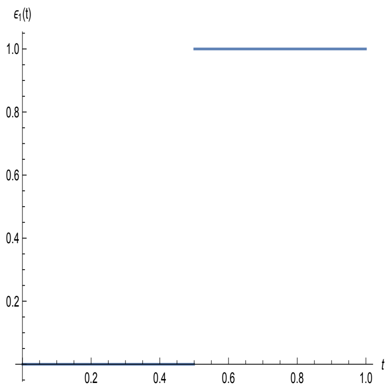

Figure 1

The introduction of the Rademacher functions $r_k(t)$, which are defined as follows, enables us to develop some rather surprising relationships out of very little:

$$r_k(t) = 1 - 2\epsilon_k(t) \tag{13}$$

for $k = 1, 2, 3, \dots$

For instance, $r_1(t)$ is set out in Figure 2.

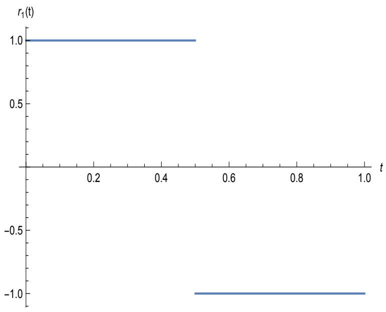

Figure 2

Note that $r_1(t) = +1$ for $0 \le t < \frac{1}{2}$ and $r_1(t) = -1$ for $\frac{1}{2} \le t < 1$

The graphs of $r_2(t)$ and $r_3(t)$ are set out below:

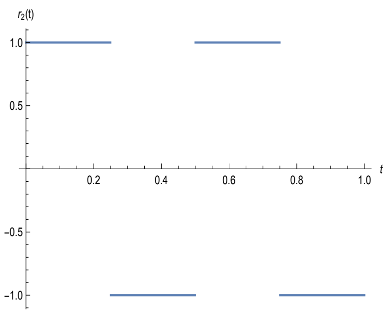

Figure 3

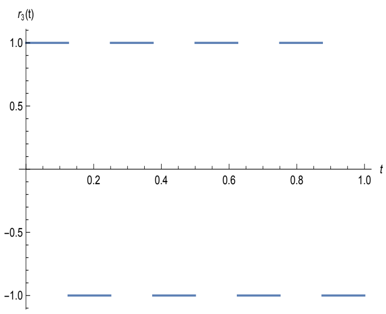

Figure 4

Using (12) and (13) the following relationship exists:

$$
\begin{aligned}
1 - 2t &= 1 - 2\sum_{k=1}^{\infty} \frac{\epsilon_k(t)}{2^k} \\
&= \underbrace{\sum_{k=1}^{\infty} \frac{1}{2^k}}_{\text{converges to 1}} \quad -2 \quad \underbrace{\sum_{k=1}^{\infty} \frac{\epsilon_k(t)}{2^k}}_{\text{converges to t}} \\
&= \sum_{k=1}^{\infty} \frac{1 - 2\epsilon_k(t)}{2^k} \\
&= \sum_{k=1}^{\infty} \frac{r_k(t)}{2^k}
\end{aligned} \tag{14}
$$

The next step in Kac's development is to note that:

$$
\int_0^1 e^{ix(1-2t)}\, dt = \frac{\sin x}{x} \tag{15}
$$

This is justified through a straightforward integration as follows:

$$
\begin{aligned}
\int_0^1 e^{ix(1-2t)}\, dt &= e^{ix} \int_0^1 e^{-2ixt}\, dt \\
&= e^{ix} \Big[ - \frac{e^{-2ixt}}{2ix} \Big]_{t=0}^{t=1} \\
&= e^{ix} \Big[ \frac{-e^{-2ix} + 1}{2ix} \Big] \\
&= \frac{-e^{-ix} + e^{ix}}{2ix} \\
&= \frac{-\cos x + i \sin x + \cos x + i \sin x}{2ix} \\
&= \frac{\sin x}{x}
\end{aligned} \tag{16}
$$

Kac also notes that:

$$
\int_0^1 e^{\left( ix \frac{r_k(t)}{2^k} \right)}\, dt = \cos \frac{x}{2^k} \tag{17}
$$

This requires a bit more work than (16). To work out the integral we need to break $[0, 1]$ into $2^k$ intervals of width $2^{-k}$ since $r_k(t)$ is a step function with oscillates between $+1$ and $-1$ over intervals of width $2^{-k}$ starting with $[0, \frac{1}{2^k})$ and ending with $(\frac{2^k-1}{2^k}, \frac{2^k}{2^k}]$. There are equal numbers of "even" and "odd" intervals, that is, an "even" interval has the form $E_j = [\frac{2j-2}{2^k}, \frac{2j-1}{2^k})$ for $j = 1, 2, \ldots, 2^{k-1}$ and an "odd" interval has the form $O_j = [\frac{2j-1}{2^k}, \frac{2j}{2^k})$ for $j = 1, 2, \ldots, 2^{k-1}$. Some things to note are that since there are $2^{k-1}$ odd and even intervals, the total number of subintervals is $2 \times 2^{k-1} = 2^k$ as it should be. Furthermore on even intervals $r_k(t) = +1$ and on

odd intervals $r_k(t) = -1$. The subintervals thus have this form:
$[0, \frac{1}{2^k}), [\frac{1}{2^k}, \frac{2}{2^k}), [\frac{2}{2^k}, \frac{3}{2^k}), \dots, [\frac{2^k-2}{2^k}, \frac{2^k-1}{2^k}), [\frac{2^k-1}{2^k}, \frac{2^k}{2^k}]$

It woulld make no difference to the integral in (17) if the last subinterval was open on the right ie excludes the single point 1.

Note that the even and odd intervals are disjoint so that we can represent the LHS of (17) as the following sum:

$$
\begin{aligned}
\int_0^1 e^{\left(ix\frac{r_k(t)}{2^k}\right)}\,dt &= \sum_{j=1}^{2^{k-1}} \underbrace{\int_{\frac{2j-2}{2^k}}^{\frac{2j-1}{2^k}} e^{\frac{ix}{2^k}}\,dt}_{r_k(t)=1 \text{ in these intervals}} + \sum_{j=1}^{2^{k-1}} \underbrace{\int_{\frac{2j-1}{2^k}}^{\frac{2j}{2^k}} e^{\frac{-ix}{2^k}}\,dt}_{r_k(t)=-1 \text{ in these intervals}} \\
&= \sum_{j=1}^{2^{k-1}} e^{\frac{ix}{2^k}} \int_{\frac{2j-2}{2^k}}^{\frac{2j-1}{2^k}} dt + \sum_{j=1}^{2^{k-1}} e^{\frac{-ix}{2^k}} \int_{\frac{2j-1}{2^k}}^{\frac{2j}{2^k}} dt \\
&= \sum_{j=1}^{2^{k-1}} e^{\frac{ix}{2^k}} \times \frac{1}{2^k} + \sum_{j=1}^{2^{k-1}} e^{\frac{-ix}{2^k}} \times \frac{1}{2^k} \\
&= \frac{1}{2^k} \sum_{j=1}^{2^{k-1}} (e^{\frac{ix}{2^k}} + e^{\frac{-ix}{2^k}}) \\
&= \frac{1}{2^k} \sum_{j=1}^{2^{k-1}} 2\cos\frac{x}{2^k} \\
&= \frac{2 \times 2^{k-1}}{2^k} \cos\frac{x}{2^k} \\
&= \cos\frac{x}{2^k}
\end{aligned}
\tag{18}
$$

Kac's next observation is the remarkable result that the integral of a product is a product of integrals:

$$
\int_0^1 \prod_{k=1}^{\infty} e^{\left(ix\frac{r_k(t)}{2^k}\right)}\,dt = \prod_{k=1}^{\infty} \int_0^1 e^{\left(ix\frac{r_k(t)}{2^k}\right)}\,dt \tag{19}
$$

To see this we start with (15), followed by (14), (10) and (18):

$$
\begin{aligned}
\frac{\sin x}{x} &= \int_0^1 e^{ix(1-2t)}\,dt \\
&= \int_0^1 e^{\left(ix\sum_{k=1}^{\infty}\frac{r_k(t)}{2^k}\right)}\,dt \\
&= \prod_{k=1}^{\infty}\cos\frac{x}{2^k} \\
&= \prod_{k=1}^{\infty}\int_0^1 e^{\left(ix\frac{r_k(t)}{2^k}\right)}\,dt
\end{aligned}
\tag{20}
$$

But:

$$
\int_0^1 e^{\left(ix\sum_{k=1}^{\infty}\frac{r_k(t)}{2^k}\right)}\,dt = \int_0^1 \prod_{k=1}^{\infty} e^{\left(ix\frac{r_k(t)}{2^k}\right)}\,dt \tag{21}
$$

So (19) follows. Is it an accident or evidence of something deeper? To investigate the matter further Kac considers the following function:

$$
\sum_{k=1}^{n} c_k r_k(t) \tag{22}
$$

The $c_k$ are constants and (22) represents a step function which is constant over the intervals $(\frac{s}{2^n}, \frac{s+1}{2^n})$ for $s = 0, 1, 2, \ldots, 2^n - 1$.

For example, consider $n = 2$. The function in (22) has the following structure:

$$
\begin{aligned}
c_1 + c_2 &\text{ on } (0, \frac{1}{4}) \\
c_1 - c_2 &\text{ on } (\frac{1}{4}, \frac{1}{2}) \\
-c_1 + c_2 &\text{ on } (\frac{1}{2}, \frac{3}{4}) \\
-c_1 - c_2 &\text{ on } (\frac{3}{4}, 1)
\end{aligned}
\tag{23}
$$

For general $n$, the function will therefore assume values of the following form:

$$
c_1 \pm c_2 \pm ... \pm c_n \tag{24}
$$

Now ignoring the constants in (23), we have 4 sequences of $+1's$ and $-1's$ ie $\{+1,+1\}, \{+1,-1\}, \{-1,+1\}, \{-1,-1\}$. This leads Kac to say that every sequence of length $n$ ( $n = 2$ in (23) ) of $+1's$ and $-1's$ corresponds to one and only one interval $(\frac{s}{2^n}, \frac{s+1}{2^n})$. On this basis we have:

$$
\begin{aligned}
\int_0^1 e^{\left(i\sum_{k=1}^{n} c_k r_k(t)\right)}\,dt &= \int_0^1 e^{\left(i\sum_{k=1}^{n} \pm c_k\right)}\,dt \\
&= \sum{}^{*} \int_{\frac{s}{2^n}}^{\frac{s+1}{2^n}} e^{\left(i\sum_{k=1}^{n} \pm c_k\right)}\,dt \\
&= \frac{1}{2^n}\sum{}^{*} e^{\left(i\sum_{k=1}^{n} \pm c_k\right)}
\end{aligned} \tag{25}
$$

It is important to note here that the symbol $\sum{}^{*}$ has a special meaning in this context: it is the sum over all the $2^n$ possible sequences of length $n$.

Kac asserts that:

$$
\frac{1}{2^n}\sum{}^{*} e^{\left(i\sum_{k=1}^{n} \pm c_k\right)} = \prod_{k=1}^{n}\left(\frac{e^{ic_k}+e^{-ic_k}}{2}\right) = \prod_{k=1}^{n}\cos c_k \tag{26}
$$

To see that this is so, note that there are $2^n$ possible n-sequences of $+1's$ and $-1's$ (with equal numbers of each sign of course) and you get a sum of exponential products that captures all these sequences and it is this symmetry which gives rise to the term $\frac{e^{ic_k}+e^{-ic_k}}{2}$ in (26). In more detail let's take $n = 2$ and track through what (26) is saying. We will have:

$$
\begin{aligned}
\frac{1}{2^n}\sum{}^{*} e^{\left(i\sum_{k=1}^{n} \pm c_k\right)} &= \frac{1}{2^2}\left[e^{ic_1+ic_2}+e^{ic_1-ic_2}+e^{-ic_1+ic_2}+e^{-ic_1-ic_2}\right] \\
&= \frac{1}{2^2}\left[e^{ic_1}\left(e^{ic_2}+e^{-ic_2}\right)+e^{-ic_1}\left(e^{ic_2}+e^{-ic_2}\right)\right] \\
&= \left(\frac{e^{ic_1}+e^{-ic_1}}{2}\right)\left(\frac{e^{ic_2}+e^{-ic_2}}{2}\right) \\
&= \prod_{k=1}^{2}\left(\frac{e^{ic_k}+e^{-ic_k}}{2}\right) \\
&= \prod_{k=1}^{2}\cos c_k
\end{aligned} \tag{27}
$$

We can complete the proof of (26) inductively:

$$
\begin{aligned}
\frac{1}{2^{n+1}}\sum\nolimits^{*} e^{\left(i\sum_{k=1}^{n+1}\pm c_k\right)} &= \frac{1}{2^{n+1}}\sum\nolimits^{*} e^{\left(i\sum_{k=1}^{n}\pm c_k \pm ic_{n+1}\right)} \\
&= \frac{1}{2^{n+1}}\left[e^{\left(i\sum_{k=1}^{n}\pm c_k + ic_{n+1}\right)} + e^{\left(i\sum_{k=1}^{n}\pm c_k - ic_{n+1}\right)}\right] \\
&= \frac{1}{2^{n}} e^{\left(i\sum_{k=1}^{n}\pm c_k\right)}\left[\frac{e^{ic_{n+1}}+e^{-ic_{n+1}}}{2}\right] \\
&= \underbrace{\prod_{k=1}^{n}\left(\frac{e^{ic_k}+e^{-ic_k}}{2}\right)}_{\text{using the obvious inductive hypothesis}} \left(\frac{e^{ic_{n+1}}+e^{-ic_{n+1}}}{2}\right) \qquad (28) \\
&= \prod_{k=1}^{n+1}\left(\frac{e^{ic_k}+e^{-ic_k}}{2}\right) \\
&= \prod_{k=1}^{n+1}\cos c_k
\end{aligned}
$$

Thus the formula is indeed true for $n+1$.

In what follows note that $\cos c_k = \int_0^1 e^{ic_k r_k(t)}\,dt$. This follows from the steps in (18) modified as follows for completeness:

$$
\begin{aligned}
\int_0^1 e^{ic_k r_k(t)}\,dt &= \sum_{j=1}^{2^{k-1}} \underbrace{\int_{\frac{2j-2}{2^k}}^{\frac{2j-1}{2^k}} e^{ic_k}\,dt}_{r_k(t)=1 \text{ in these intervals}} + \sum_{j=1}^{2^{k-1}} \underbrace{\int_{\frac{2j-1}{2^k}}^{\frac{2j}{2^k}} e^{-ic_k}\,dt}_{r_k(t)=-1 \text{ in these intervals}} \\
&= \sum_{j=1}^{2^{k-1}} e^{ic_k}\int_{\frac{2j-2}{2^k}}^{\frac{2j-1}{2^k}} dt + \sum_{j=1}^{2^{k-1}} e^{-ic_k}\int_{\frac{2j-1}{2^k}}^{\frac{2j}{2^k}} dt \\
&= \sum_{j=1}^{2^{k-1}} e^{ic_k}\times\frac{1}{2^k} + \sum_{j=1}^{2^{k-1}} e^{-ic_k}\times\frac{1}{2^k} \qquad (29) \\
&= \frac{1}{2^k}\sum_{j=1}^{2^{k-1}}(e^{ic_k}+e^{-ic_k}) \\
&= \frac{1}{2^k}\sum_{j=1}^{2^{k-1}} 2\cos c_k \\
&= \frac{2\times 2^{k-1}}{2^k}\cos c_k \\
&= \cos c_k
\end{aligned}
$$

Thus we have:

$$\int_0^1 e^{\left(i\sum_{k=1}^n c_k r_k(t)\right)}\,dt = \prod_{k=1}^n \cos c_k = \prod_{k=1}^n \int_0^1 e^{ic_k r_k(t)}\,dt \tag{30}$$

Now put $c_k = \frac{x}{2^k}$ in (30):

$$\int_0^1 e^{\left(ix\sum_{k=1}^n \frac{r_k(t)}{2^k}\right)}\,dt = \prod_{k=1}^n \cos\frac{x}{2^k} \tag{31}$$

Kac then notes that $\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{r_k(t)}{2^k} = 1-2t$ uniformly on $(0,1)$. This point is essental for the interchange of summation and integration that occurs below. To prove this assertion one can use the Weiestrass M-test as follows. Since:

$$\left|\frac{r_k(t)}{2^k}\right| \leq \frac{1}{2^k} = M_k \text{ for all } k = 1, 2, \ldots \tag{32}$$

and $\sum_{k=1}^{\infty} M_k = 1$ it follows that $\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{r_k(t)}{2^k}$ does converges uniformly to $1-2t$ on $(0,1)$.

Thus we have:

$$\begin{aligned}
\frac{\sin x}{x} &= \int_0^1 e^{ix(1-2t)}\,dt = \int_0^1 e^{\left(ix\lim_{n\to\infty}\sum_{k=1}^{n}\frac{r_k(t)}{2^k}\right)}\,dt \\
&= \lim_{n\to\infty}\int_0^1 e^{\left(ix\sum_{k=1}^n \frac{r_k(t)}{2^k}\right)}\,dt \\
&= \lim_{n\to\infty}\prod_{k=1}^{n}\cos\frac{x}{2^k} \\
&= \prod_{k=1}^{\infty}\cos\frac{x}{2^k}
\end{aligned} \tag{33}$$

Thus equation (10) has been proved by a different route.

## 3 What is the property of the binary digits that makes the proof tick?

There is something about the functions $r_k(t)$ which captures properties of binary digits and Kac's next step is to lay the foundation for a probabilistic connection. He considers the set of $t's$ for which:

$$r_1(t) = +1, \quad r_2(t) = -1, \quad r_3(t) = -1 \tag{34}$$

By looking at Figures 2-4 it is clear that this set is the interval $(\frac{3}{8}, \frac{4}{8})$ (ignoring end points which, in the final analysis, will not make any difference). Now the length of this interval is simply $\frac{1}{8}$ and :

$$\frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \tag{35}$$

While this looks trivial it is actually describing the measure (length) of the required set as a product of disjoint sub-intervals whose length is $\frac{1}{2}$. When (35) is written in standard measure-theoretic terms ( where $\mu(X)$ is the measure (length) of set X ) as follows what is going on becomes clearer:

$$\mu\{r_1(t) = +1, r_2(t) = -1, r_3(t) = -1\} = \mu\{r_1(t) = +1\}\, \mu\{r_2(t) = -1\}\, \mu\{r_3(t) = -1\} \tag{36}$$

Clearly (36) generalises to an arbitrary number of Rademacher functions.

If $\delta_1, \delta_2, \ldots, \delta_n$ is a sequence of $+1's$ and $-1's$ then:

$$\mu\{r_1(t) = \delta_1, r_2(t) = \delta_2, \ldots, r_n(t) = \delta_n\} = \mu\{r_1(t) = \delta_1\}\, \mu\{r_2(t) = \delta_2\} \ldots \mu\{r_n(t) = \delta_n\} \tag{37}$$

which looks suspiciously like:

$$\left(\frac{1}{2}\right)^n = \underbrace{\frac{1}{2} \times \frac{1}{2} \times \ldots \frac{1}{2}}_{n \text{ times}} \tag{38}$$

Using this insight (31) can be proved as follows:

$$\begin{aligned}
\int_0^1 e^{\left(i \sum_{k=1}^n c_k r_k(t)\right)}\, dt &= \sum_{\delta_1, \ldots, \delta_n} e^{\left(i \sum_{k=1}^n c_k \delta_k\right)}\, \mu\{r_1(t) = \delta_1, \ldots, r_n(t) = \delta_n\} \\
&= \sum_{\delta_1, \ldots, \delta_n} \prod_{k=1}^n e^{i c_k \delta_k} \prod_{k=1}^n \mu\{r_k(t) = \delta_k\} \\
&= \sum_{\delta_1, \ldots, \delta_n} \prod_{k=1}^n e^{i c_k \delta_k} \mu\{r_k(t) = \delta_k\} \\
&= \prod_{k=1}^n \sum_{\delta_k} e^{i c_k \delta_k} \mu\{r_k(t) = \delta_k\} \\
&= \prod_{k=1}^n \int_0^1 e^{i c_k r_k(t)}\, dt
\end{aligned} \tag{39}$$

There are some comments that need to be made about the steps in (39). This step in particular requires comment:

$$\int_0^1 e^{\left(i\sum_{k=1}^{n} c_k r_k(t)\right)}\, dt = \sum_{\delta_1,\dots,\delta_n} e^{\left(i\sum_{k=1}^{n} c_k\delta_k\right)}\, \mu\{r_1(t)=\delta_1,\dots,r_n(t)=\delta_n\} \tag{40}$$

The LHS of (40) is as follows, where for convenience, $n = 3$ and it is assumed that $\delta_1 = +1, \delta_2 = -1, \delta_3 = -1$.

$$\begin{aligned}\sum_{k=1}^{2^3}\int_{\frac{k-1}{2^3}}^{\frac{k}{2^3}} e^{\left(i\sum_{k=1}^{3} c_k r_k(t)\right)}\, dt = {} & \frac{1}{8}e^{i(c_1+c_2+c_3)} + \frac{1}{8}e^{i(c_1+c_2-c_3)} + \frac{1}{8}e^{i(c_1-c_2+c_3)} + \frac{1}{8}e^{i(c_1-c_2-c_3)} \\ & + \frac{1}{8}e^{i(-c_1+c_2+c_3)} + \frac{1}{8}e^{i(-c_1+c_2-c_3)} + \frac{1}{8}e^{i(-c_1-c_2+c_3)} + \frac{1}{8}e^{i(-c_1-c_2-c_3)}\end{aligned} \tag{41}$$

The $r_k(t)$ with the highest index governs the fineness of each step function to integrate over and that is where the $\frac{1}{8}$ factor comes from since $r_3(t)$ oscillates between $+1$ and $-1$ on $(\frac{k-1}{8}, \frac{k}{8})$ for $k = 1, 2, \dots, 8$. A bit of algebra shows that (41) equals $\cos c_1 \cos c_2 \cos c_3$ in accordance with (30).

The RHS of (40) requires some interpretation. The sum $\sum_{\delta_1,\dots,\delta_n}$ is a sum over the $2^n$ sequences of $\delta_1, \delta_2, \dots, \delta_n$ and each such sequence is represented in the exponential term $e^{\left(i\sum_{k=1}^{n} c_k\delta_k\right)}$. The factor $\mu\{r_1(t) = \delta_1, \dots, r_n(t) = \delta_n\}$ reflects the fact that in performing the integration one looks for the interval that meets the simultaneous conditions $r_1(t) = \delta_1, \dots, r_n(t) = \delta_n$ over which to perform the integration of the step function for each combination of $\delta_1, \delta_2, \dots, \delta_n$. Because we are dealing with step functions this summation process does indeed equal the integral on the LHS of (40). In the case of $n = 3$ and with our assumptions about the $\delta_k$, $\mu\{r_1(t) = \delta_1, r_2(t) = \delta_2, r_3(t) = \delta_3\} = \frac{1}{8}$. Thus:

$$\begin{aligned}\sum_{\delta_1,\dots,\delta_n} e^{\left(i\sum_{k=1}^{n} c_k\delta_k\right)}\, \mu\{r_1(t)=\delta_1,\dots,r_n(t)=\delta_n\} = {} & \frac{1}{8}e^{i(c_1+c_2+c_3)} + \frac{1}{8}e^{i(c_1+c_2-c_3)} + \frac{1}{8}e^{i(c_1-c_2+c_3)} \\ & + \frac{1}{8}e^{i(c_1-c_2-c_3)} + \frac{1}{8}e^{i(-c_1+c_2+c_3)} + \frac{1}{8}e^{i(-c_1+c_2-c_3)} + \frac{1}{8}e^{i(-c_1-c_2+c_3)} + \frac{1}{8}e^{i(-c_1-c_2-c_3)}\end{aligned} \tag{42}$$

Equation (37) justifies the second and third lines in (39) and encapsulates indepedence. The interchange of the summation and product in the fourth line is justified by the fact that the sum is finite and we are dealing with step functions. The final step where the sum is equated with the integral follows the same logic as the first step but in reverse ie moving from the sum to the integral.

## 4 The probabilistic connection

Equation (37) reeks of probability for we know that if events $A_1, A_2, \dots, A_n$ are independent then:

$$\text{Prob }\{A_1 \text{ and } A_2 \text{ and } A_3 \text{ and } \dots A_n\} = \text{Prob }\{A_1\}.\text{Prob }\{A_2\}\dots\text{Prob }\{A_n\} \tag{43}$$

Thus the probabiity of any sequence of length $n$ of tosses of a fair coin is $\frac{1}{2^n}$.

Kac associates the model for coin tossing with the Rademacher functions $r_k(t)$ as follows:

Head (H) $\leftrightarrow +1$

Tail (T) $\leftrightarrow$ -1

$k^{th}$ toss $(k = 1, 2, \dots) \leftrightarrow r_k(t)$ $(k = 1, 2, \dots)$

Event $\leftrightarrow$ Sets of $t's$

Probability of an event $\leftrightarrow$ Measure of the corresponding set of $t's$

Kac then considers the problem of finding the probability that in $n$ independent tosses of a fair coin, exactly $l$ will be heads. Using the above correspondence this problem is converted to finding the measure of the set of $t's$ such that exactly $l$ of the $n$ numbers $r_1(t), r_2(t), \dots, r_n(t)$ are equal to $+1$.

This is equivalent to:

$$r_1(t) + r_2(t) + \dots + r_n(t) = l + (n - l) \times -1 = 2l - n \tag{44}$$

since $n - l$ of $r_k(t)$ must be $-1$.

Now if $m$ is an integer then:

$$\frac{1}{2\pi}\int_0^{2\pi} e^{imx}\, dx = \begin{cases} 1, & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases} \tag{45}$$

This is straightforward: if $m = 0$ the integral is simply $\frac{1}{2\pi} \times 2\pi = 1$ and of $m \neq 0$ it is $\frac{1}{2\pi im}(e^{2\pi im} - 1) = \frac{1}{2\pi im}(\cos 2\pi m - 1) = 0$.

Therefore:

$$\phi(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix[r_1(t)+\dots+r_n(t)-(2l-n)]}\, dx = \begin{cases} 1, & \text{if (44) holds} \\ 0, & \text{otherwise} \end{cases} \tag{46}$$

Next Kac states that the measure of the required set and hence the relevant probability is as follows:

$$\mu\{r_1(t) + r_2(t) + \dots + r_n(t) = 2l - n\} = \int_0^1 \phi(t)\, dt \tag{47}$$

The LHS of (47) is the measure of the relevant set of $t's$ while on the RHS of (47) we have to integrate over $[0,1]$ to capture all the instances of the sets of $t$ such that $l$ out of $n$ of the $r_k(t)$ are equal to 1.

The next step is to perform the integration of the RHS of (47):

$$\begin{aligned}\mu\{r_1(t)+r_2(t)+\cdots+r_n(t)=2l-n\} &= \int_0^1 \phi(t)\,dt \\ &= \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} e^{ix[r_1(t)+\cdots+r_n(t)-(2l-n)]}\,dx\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(2l-n)x}\left(\int_0^1 e^{ix[r_1(t)+\cdots+r_n(t)]}\,dt\right)dx\end{aligned} \tag{48}$$

As Kac points out, the interchange of the order of integration in the last step does not require Fubini's theorem since we are dealing with step functions in the form of $r_1(t)+\cdots+r_n(t)$

Now in (30) let $c_1=c_2=\cdots=c_n=x$ so that $\int_0^1 e^{ix\sum_{k=1}^n r_k(t)}\,dt = \prod_{k=1}^n \cos x = \cos^n x$ and hence (noting that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$):

$$\begin{aligned}\mu\{r_1(t)+r_2(t)+\cdots+r_n(t)=2l-n\} &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(2l-n)x}\,\cos^n x\,dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(2l-n)x}\,\frac{1}{2^n}\sum_{k=0}^n\binom{n}{k}e^{ikx}e^{-i(n-k)x}\,dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2^n}\sum_{k=0}^n\binom{n}{k}e^{ikx-i(n-k)x-i(2l-n)x}\,dx \\ &= \frac{1}{2^{n+1}\pi}\int_0^{2\pi}\sum_{k=0}^n\binom{n}{k}\underbrace{e^{2i(k-l)x}}_{1\text{ when }k=l,\text{ otherwise }0}\,dx \\ &= \frac{1}{2^{n+1}\pi}\times 2\pi\binom{n}{l} \\ &= \frac{1}{2^n}\binom{n}{l}\end{aligned} \tag{49}$$

From basic probability theory the required probability is that of choosing exactly $l$ out of $n$ of the $2^n$ equally likely sequences ie $\frac{1}{2^n}\binom{n}{l}$.

Kac's book goes on to develop many other deep themes and is worth reading from cover to cover. For more detail on some of the properties of Rademacher functions see [2].

## 5 References

[1] Mark Kac, *"Statistical Independence in Probability, Analysis and Number Theory"*, The Mathematical Association of America, 1964. The book can be accessed via this link: `http://www.gibbs.if.usp.br/~marchett/estocastica/MarkKac-Statistical-Independence.pdf`

[2] Peter Haggstrom, "A combinatorial view of the orthogonality of Rademacher functions ", `https://www.gotohaggstrom.com/A%20combinatorial%20view%20of%20the%20orthogonality%20of%20Rademacher%20functions.pdf`

## 6 History

Created 05 September 2018//

06 September 2018 - link to Carus book added