Relationship between the characteristic function and Fourier transforms.

The characteristic function is defined by :

$$\phi_X(\xi) = E\left[e^{i\xi X}\right]$$

The Fourier transform of a function (x) is defined as (there are other equivalently scaled definitions) as :

$$F(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{i\pi\xi x}\, dx$$

In the contextof stochastic processes,
if we take f (x) as the Gaussian function we get the fundamental result that the the characteristic function is equal to the function itself (subject to a scaling factor). This is the same as saying that the Fourier transform of the Gaussian function is itself.

In what follows take $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

We need to evaluate $\phi_X(\xi) = E\left[e^{i\xi X}\right]$

The answer is $e^{-\xi^2/2}$

So CF $\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right) \rightarrow e^{-\xi^2/2}$ (see for instance Feller Vol 2 page 503)

A student ince asked where the minus sign came from. There is no simplistic answer to this. You either have to do the integral or implicitly rely upon properties of Fourier transforms which effectively amounts to the same thing.

By definition

$$E\left[e^{i\xi X}\right] = \int_{-\infty}^{\infty} f(x)\, e^{i\xi x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} . e^{i\xi x}\, dx \quad \ldots\ldots\ldots\ldots (*)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 + 2i\xi x)/2}\, dx$$

Let $y = x - i\xi$, then $dy = dx$ for fixed ξ

Then $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 + 2i\xi x)/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\xi}^{\infty - i\xi} e^{-(y^2 - 2i\xi x - \xi^2 + 2i\xi x)/2}\, dy$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\xi}^{\infty - i\xi} e^{-(y^2 - 2i\xi x - \xi^2 + 2i\xi x)/2}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\xi}^{\infty - i\xi} e^{-(y^2 - \xi^2)/2}\, dy$$

$$= e^{\xi^2/2}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty - i\xi}^{\infty - i\xi} e^{-y^2/2}\, dy$$

We now have to evaluate this integral $\int_{-\infty - i\xi}^{\infty - i\xi} e^{-y^2/2}\, dy$

This involves complex integration theory in order to give a rigorous analytic proof. Because this is such a fundamental result it is worth doing once.

Note that when $\xi = 0$ we get that $E[1] = \int_{-\infty}^{\infty} f(x)\, dx = 1$

We have to integrate around a "toy contour". Cauchy's Theorem is the essence of this process $e^{-x^2 i/2}$ is analytic (or "holomorphic") on the interior of the contour and hence Cauchy's Theorem applies.

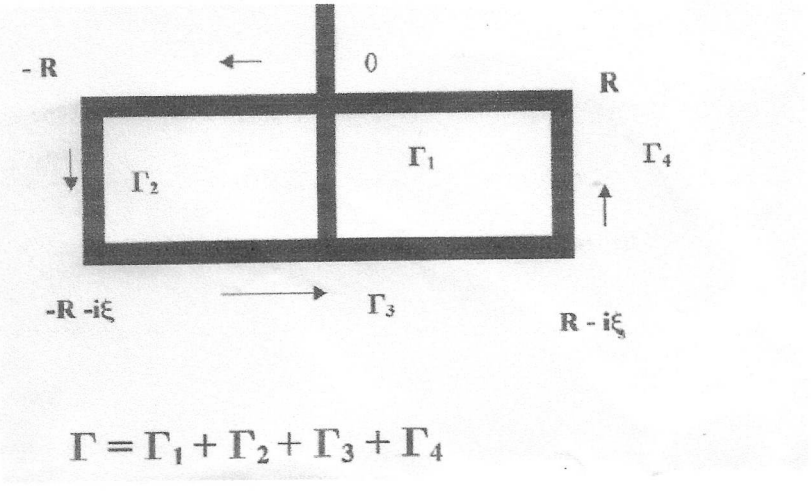
In fact e^z. cos z, sin z etc (ie power series in disguise) are analytic everywhere. If you have not done complex variable theory you will need to consult any standard text. Suffice it to say that Cauchy's Theorem guarantees that the integral of $e^{-x^2 i/2}$ around a closed contour is zero.

If f (z) is analytic on the interior of the closed contour,

$\int_{\text{Closed contour}} f(z)\, dz = 0 \rightarrow$ Cauchy's Theorem

You may wonder where getting a zero result helps. The answer is that you decompose the closed contour integral into disjoint segments, and hence integrals, and sum them up, paying attention to orientation of the path (we go counter - clockwise so the interior is on the inside). Indeed, one of the great attractions of complex variable theory is that seemingly intractable real valued integrals can be solved very easily by extending the integration into the complex domain and gettng the benefit of some very powerful results.

Here is the "toy" contour :



Integral around Γ_1 : Along this path note that ξ runs from R to - R) we are dealing just with real values, say x.

$$\int_{\Gamma_1} f(z)\,dz = \lim_{R\to\infty} \int_R^{-R} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = -\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = -1$$

Integral around Γ_2 : along this path $z = -R - iy$ and $dz = -i\,dy$

$$\int_{\Gamma_2} f(z)\,dz = \int_0^{\xi} e^{-(-R-iy)^2/2}\,(-i\,dy)$$

$$= -i \int_0^{\xi} e^{-(R+iy)^2/2}\,dy$$

$$= -i \int_0^{\xi} e^{-(R^2+2\,iRy-y^2)/2}\,dy$$

The integral on the right converges due to the following estimates :

$$\left| -i \int_0^{\xi} e^{-(R^2+2\,iRy-y^2)/2}\,dy \right| \le \int_0^{\xi} \left| e^{-(R^2+2\,iRy-y^2)/2}\,dy \right|$$

$$\le \int_0^{\xi} \left| e^{-(R^2+2\,iRy-y^2)/2} \right| dy$$

$$\le e^{-R^2/2} \int_0^{\xi} \left| e^{-iRy} \right| \left| e^{y^2/2}\,dy \right.$$

$$= e^{-R^2/2} \int_0^{\xi} e^{y^2/2}\,dy \qquad \text{since } \left| e^{-iRy} \right| = 1 \;\forall\, R, y$$

$$\le \xi\, e^{-R^2/2}\, e^{\xi^2/2} \quad \text{since } e^{y^2/2} <= e^{\xi^2/2} \;\forall\, y \in [0, \xi]$$

So $\exists\, C > 0$, $\ni$ $\left| -i \int_0^{\xi} e^{-(R^2+2\,iRy-y^2)/2}\,dy \right| <= Ce^{-R^2/2}$ **where C depends on fixed ξ**

Since $Ce^{-R^2/2} \to 0$ **as** $R \to \infty$, $\int_{\Gamma_2} f(z)\,dz = 0$

By a symmetrical argument $\int_{\Gamma_4} f(z)\,dz = 0$

This leaves the integral along Γ_3. Along Γ_3 $z = x - i\xi$ where ξ constant
Hence dz = dx

$$\int_{\Gamma_3} f(z)\,dz = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} e^{-(x^2-i\xi)^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} e^{-(x^2-2\,ix\xi-\xi^2)^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\xi^2/2} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{i\xi x}\,dx$$

Putting all the bits together we have that :

$$\int_{\Gamma} f(z)\,dz = 0$$

But $\int_{\Gamma} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_3} f(z)\,dz + \int_{\Gamma_4} f(z)\,dz$

$$0 = -1 + \frac{1}{\sqrt{2\pi}} e^{\xi^2/2} \int_{-R}^{R} e^{-x^2/2}\, e^{i\xi x}\,dx$$

$$\text{So } \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-x^2/2}\, e^{i\xi x}\,dx = e^{-\xi^2/2} \quad \ldots\ldots\ldots\ldots(**)$$

Recall from $(*)$ **that** $E\left[e^{i\xi X}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}.e^{i\xi x}\,dx$

Thus we have shown that $E\left[e^{i\xi X}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}.e^{i\xi x}\,dx$

$$= e^{-\xi^2/2} \quad \text{(from (**))} \quad \text{for } \xi>=0$$

By symmetry in the upper half plane we will get the same result when $\xi<0$

$$\text{So } \forall\xi,\quad E\left[e^{i\xi X}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}.e^{i\xi x}\,dx$$

$$= e^{-\xi^2/2}$$

There are other ways of proving this result but this is a rigorous proof that one would do in a course on Fourier Theory. Note that the modern proofs of the Central Limit Theoren have been based on characteristic functions.

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